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JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. XVII.

LONDON:
CHARLES & EDWIN LAYTON, 150 FLEET STREET.
NEW YORK: J. H. & C. M. GOODSELL, 156 BROADWAY; AND 156 & 157 WASHINGTON STREET, CHICAGO.
PARIS: ARMAND ANGER, LIBRAIRIE DES ASSURANCES, 48 RUE LAFFITTE.
HAMBURG: W. MAUKE SÖHNE. LA HAGUE: MARTINUS NIJHOFF.

1873.

LONDON:
PRINTED BY CHARLES AND EDWIN LAYTON,
FLEET STREET AND GOUGH SQUARE.

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On the Valuation of Claims upon current Policies in the Liquidation of a Life Office, with reference to the Decisions in Bell's and Lancaster's Cases. By C. J. BUNYON, M.A., *Barrister-at-Law, a Vice-President of the Institute of Actuaries.*

[Read before the Institute, 29th January 1872.]

THE question which it is proposed to consider in the present paper is the correct mode of valuation of current policies, or true measure of proof to which claimants thereon are entitled in the liquidation of a Life Office. The subject is of especial interest at the present time, because its solution is even now waiting settlement by the Court of Chancery; and the interest is heightened by the fact, that conflicting judgments have been delivered by two eminent equity Judges, namely, by Lord Justice James, in Bell's case, when sitting as Vice-Chancellor in the Court of Chancery, and by Lord Cairns, in Lancaster's case, when sitting as Arbitrator in the Albert Arbitration. It does not appear that among actuaries there is a complete agreement. Nevertheless, the case ought not to present any great difficulty, and whether among actuaries or in a Court of Equity must be decided by argument alone. It is therefore proposed to consider the question, in the first instance, upon principle; and secondly, with reference to the doctrines laid down in the cases to which we have referred.

It will, of course, be remarked by every lawyer at the outset of

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another mode of speaking of the market price of the property to be valued, since here we find the only mart for the sale *de novo* of insurances on lives.

As a matter of abstract justice, then, when you have ascertained the market price of the insurable life, there does not appear any sufficient reason why you should not allege, and prove, if you can, the additional value arising from uninsurability. This is not a collateral damage personal to the claimant, but arises upon the policy itself, which would be undoubtedly appreciated thereby. Everybody knows that, as a matter of fact, the health of the assured is the principal element to be considered in valuing a single policy, and that a dangerous accident or illness, which has prostrated the strength beyond recovery, may so enhance the value as almost to reduce the insurance money into possession. The true objection to permitting proof of uninsurability appears to be the difficulty of proof which must rest upon opinion alone, not of a rival Office, but of medical witnesses for the claimant, or, at least, of an independent referee. It may well happen that the cost, trouble, and uncertainty of the proof may justify its rejection; but if so, it is submitted that it must be rejected upon the very words of the statute, which requires a just estimate *as far as is possible* of the value.

Leaving this point then for the decision of the Court of Chancery, we come to the general question of the measure of value or terms of valuation, on the assumption that all the lives are of an average value. And here the limits of valuation, for the purpose of this present discussion, appear to be found in what are known to actuaries as pure or net and gross premium valuations respectively. It does not of course follow that either is to be accepted as necessarily correct, and it may well happen that the true measure is to be found in some mean between the two. It is also here proper to premise that in all these liquidations we have to determine the relative shares of individual claimants in a limited fund, since, as a rule, the rights of claimants, in consequence of the form in which policies are all but universally issued, are limited to the funds and capital alone. It has been decided over and over again that profit policyholders are not co-partners; that they, except in a Mutual Insurance Office, separately contract with the Company alone; and that they have no express privity of contract with each other. But, on the other hand, Courts of Equity recognise that in one sense mutual insurance is at the base

(7.) And lastly, since the policyholder is entitled to bring in

of every Insurance Company, and that there is a *quasi* co-partnership, at least, in interest; and hence, upon a liquidation, all must be held to be affected by any general principles of equity which we can discover as governing the subject.

The argument in favour of the pure or net premium valuation appears to take the following form. "The assumed premium payable"—the words are those of the arbitrator—"is divisible into "two parts. The first, the part which it is calculated will provide "for the risk, called the pure premium; and secondly, the addition "for Office expenses and other charges, which is sometimes called "the loading. The pure premium only is to be taken into account." It is commonly argued that this loading in an Office properly managed is never valued, or a reserve is made which is an equivalent to it, and hence the sum which the Office ought to have in hand for every policy is the reversionary value of the sum assured, less the value of the pure premium only. If then this is the sum which the Office ought to have in hand upon sound principles of finance, it is surely, it is said, the measure of damage to the assured. Again, it is assumed that every policyholder is a loser by the insolvency although he may have paid but a single premium, and that every policy must have a value; and it is argued that it is by a pure premium valuation alone, or its equivalent, that this can be the case. By valuing the gross premiums, it is urged, recent policies become not liabilities, but assets; and it is absurd when you are seeking the measure of damage for an admitted wrong to be told—and this would be the result of a gross premium valuation—that you had suffered nothing, had no cause of complaint, and were to think yourself very lucky if you got off with nothing more to pay. It is also thought that as between profit and non-profit policies, the system is a just one, for the value of the risks is reserved apart from the question of loading, and the profit policyholder losing the excess which he has paid in past years, claims equally with the non-profit policyholder upon the insurers. He sacrifices the excess of loading which he has paid, for the hope of profits in the past; and since the prospect of profits for the future has passed away, and is absolutely lost, he pays such excess no longer, and in all things there is an equality. There are also arguments in favour of this system, on the consideration that it admits of one scheme of valuation to the extreme limit of life, and is therefore less arbitrary at the higher ages than what is called the reinsurance system; that it may be applied without inconvenience, whatever the

principle on which the tables of premiums have been constructed; that it estimates with an equal measure the liabilities of all Companies, and is—but this last is I think here a worthless argument—in common use with very many. It must not however be supposed that it is used universally. It is only one mode of valuing out of many, and, in proof of its partial acceptance, is not adopted by the Equitable, Law Life, Atlas, Rock, Amicable, London Life, and many other foremost Companies. The writer does not doubt but that it is an admirable system for a going concern. The contention is that it is not of universal application, and is, for reasons which we shall next consider, inappropriate upon an insolvency.

If this is a fair statement of the argument in favour of the pure premium system of proof, we will now proceed to consider the objections to it. These are—

(1.) That it depends entirely upon an arbitrary assumption which has no existence out of the mind of the actuary. It would be imagined by a layman, from the language of the arbitrator, that the pure premium was a distinct entity, well known and readily distinguished, and to be valued by the 17 Offices' Table; but actuaries are aware that it is the mere child of an hypothesis and an afterthought, the device of Epimetheus and not of Prometheus, represented by one scale under the 17 Offices' Experience, by another under the Carlisle, and so on through the whole list of actuaries' tables.

(2.) That there is no contract, express or implied, for its use, and that it is probably inconsistent with the past theory and practice of the Company. If it can be proved that the tables were constructed in the manner suggested, by ascertaining the pure premiums and adding a loading to them, and that a pure premium valuation was part of the contract, there would be an end of all controversy; but in fact it can rarely, if ever, be proved that tables were so constructed, or that, if they were, there was any partnership rule or express or implied agreement for a pure premium valuation for the future. In the Albert case the elements of the pure premium valuation were settled to be the 17 Offices' Experience and 4 per cent. interest, but it is almost certain that not one of the amalgamated Companies, either in the Albert or European, had their tables of rates calculated upon any such base, and still more so, that there was in no case any express or implied contract for a pure premium valuation upon it in the conduct of the business. The rule then is wholly arbitrary unless it can be supported upon some

irrefragible scientific principle, which is violated by a departure from it.

(3.) The reservation of the entire loading in the case of an insolvency is unnecessary and inconsistent with the object of its imposition, which is to provide for expenses, commissions, fluctuations and profits. The principle of the pure premium valuation is to reserve this fund intact for the future. This may be sound finance for a going concern, but is it necessary or proper for an insolvent one? We admit as of course that a provision must be made for expenses, but commissions now cease to be payable. The loading is added for fluctuations, and the fluctuations have happened; why should it not be applied to the very purpose for which it was intended? In very recent policies, it may be said, you cannot count upon it as you cannot compel the payment of premiums for the future, but in a pure premium valuation the Office debits itself with the entire capitalized value of this fluctuation fund, and by so doing alone brings out its liability. A still further portion of the loading, indeed the whole, if we take it to measure the bonus expectations of the public, is intended for profits. But here the argument laid down in Bell's case comes with crushing force. The assured contracted for profits, and he contracted during life to pay an advanced rate above the non-profit scale for the privilege of sharing in the profits. He has made a bad bargain, for there are no profits; but because the speculation has turned against him, he cannot complain and ask for an alteration of the terms of his contract, that is to say, that his policy should be calculated as if he had agreed to pay a lower rate of premium only.

(4.) The pure premium valuation has, then, in our particular case, this inherent vice, that it alters the contract contained in the policy. That, as regards the future, it reduces the premium which the assured has agreed to pay, and gives him a benefit at the expense of the other classes of insurers or creditors who look for payment to a limited fund, or for a dividend out of it. The fact that the assured has in past time paid an increased premium seems to avail nothing, for this was part of his contract; and here it is to to be observed, that up to the time of the stoppage an insolvent Office does generally give bonuses. These, as a rule, cannot be rescinded, and the insured has had all that he bargained for. The truth is, that such part of the loading as can be proved to have been added for the purpose of giving profits belongs to the creditor, and by striking it out of the calculation he is defrauded. But it is said the claimant, even if a policyholder of the most recent date, is

entitled to something—his policy must have some value. Why so? If you assume the rates of mortality and interest proposed as the true base of calculation, why should you not say that the bonus policyholder agreed, in exchange for his share of the profits, to pay such a premium as would for a certain number of years prevent his policy having any surrender or Office value, when compared with a non-profit policy. This is what in fact he did, and if he makes a bad speculation and gets no bonus, why attempt to alter facts?

(5.) Again, the Office is liable for the present value of the sums assured, and is entitled to take credit for the present value of the premiums calculated as annuities. It is also liable to the payment of annuities. If it is to pay the latter annuities in full, why should it be entitled to take credit for only a proportion of say $\frac{4}{5}$ths of the former annuities—that is, the premiums. This alone may be argued, in the words of the Arbitrator, "to present "an insuperable difficulty," viz., "that the pure premium rule is "not homogeneous with the mode adopted by the Court of Chancery "in a winding up of valuing annuities."

(6.) Again, it is admitted in all the arguments that the ground of complaint of the life still insurable is, that he must pay a higher premium than before, if he goes to another Office. The difference between the two rates is then the measure of his loss, and its value is familiar to us all in the common expression

$$(P_{x+n}-P_x)(1+a_{x+n}).$$

If we denote the pure premium by P′, the formula for a pure premium valuation is also

$$(P'_{x+n}-P'_x)(1+a_{x+n}),$$

but in no case does the second expression coincide with the first, except in the solitary one in which the loading is a constant at all ages, and even then the coincidence is in form only, for the values are very different—the former representing the entire reservation, the second omitting the value of the loading, which is not the less reserved, and, if we denote the constant by c, is represented by the expression $c(1+a_{\overline{x+n}})$. The pure premium valuation does not then appear mathematically just. This is, however, a subject which has already been discussed with great learning and ability in the *Journal of the Institute of Actuaries*,* and I therefore refrain from pursuing it further in the present paper.

* See vol. x p. 312, and Mr. Sprague's recent admirable paper on the Liquidation of an Insolvent Company, which is in itself a complete refutation of the proposal to apply a pure premium estimate in such a case.

his claim and prove for the value of the sum assured giving credit for the value of the premiums which he has to pay—to give him more than this is to entitle him *to prove for profits*—which is absurd. This position is most completely illustrated by the case of policies on which the annual premium has been commuted for a single payment. Imagine the entrance into a Society of two persons on the same day and at the same age, insuring for the same amounts, on the payment of single premiums, but the one upon the profit and the other upon the non-profit scale. Upon the insolvency of the Office the interests of these persons, assuming no previous bonus allotted to the first, are precisely equal—the claim in both cases is for the value of the sum assured alone—but the one has evidently paid a much larger consideration than the other, and the difference between the profit and non-profit premiums is evidently the purchase money of a share in the future profits of the concern. If you make any distinction in favour of the first you clearly entitle him to prove for his loss of profits, and if you do not, and yet adopt a pure-premium valuation for policies still subject to annual payments, you make an unjust distinction in favour of the latter. It is not unfair to contend that we have here brought our case to this simple and tangible issue, namely, whether a person, who, without being himself a partner, has purchased a share in the future profits of a partnership which has become insolvent, can prove in the bankruptcy for the value of his lost expectations, a proposition which will not, of course, bear argument for an instant.

The decision in Lancaster's case is really a singular one when coming from so eminent a lawyer as Lord Cairns, of whom we would desire to speak with all possible respect, but it is explained by the fact that he was guided by the opinion of experts, "fortified by the best advice which he could procure from the best authorities," and this advice seems to have been hopelessly misleading and erroneous. Indeed, in the last paragraph of his judgment, the legal mind of the arbitrator appears to reject it, for he observes:—"The observations which I have made" (p. 683) "put "out of the case altogether the question of any peculiar mode of "valuation of profit policies. That I consider to have been properly "dealt with by the Court of Chancery. The portion of the premium "attributable to the sharing of the profits is altogether out of the "case." It is indeed put out of the case by the pure premium valuation system, for the profit premium is thereby reduced at once to the same scale as the non-profit, but this, we submit, was not by any means the intention of the Court of Chancery.

Can it then be contended that a gross premium valuation is admissible? To this question the answer must be a negative, because there is then no provision for the necessary expenses of carrying on or rather winding up the business. And here the observations of Lord Justice James, when Vice-Chancellor (*in re* the European, 9 Eq. Ca., 130), may be read with much instruction. He addresses himself to this very question. It had been argued on the evidence of the actuaries that the loading must be struck off from the premiums in valuation, or at least a loading equal to the then current expenses, which, marvellous to relate, amounted to a full loading of nearly 25 per cent., or £70,000 out of an income of £290,000, and that the value of the loading must accordingly be added to the liability arising upon the gross valuation. "If" said the Vice-Chancellor, "I was to take upon myself to appoint a "manager as receiver of the premiums due from the policyholders, "they would all have to pay their money, for that is in their "contract, without putting the Company to any further expense. "Then I do not think that there is that inaccuracy in the estimate "of the liabilities which was first pressed upon me, and I cannot "add to it that three-quarters of a million, or any great portion "of it, in testing the question of solvency." This is quite apart from the question whether the European was insolvent, and we now know that it was so; but however strong the presumption of insolvency was, a presumption equal in the mind of an actuary to a certainty, the Court had no alternative but to dismiss that petition. I thought at the time that the Petitioner might have succeeded in obtaining redress upon a Bill properly framed, but this question is now superannuated by "The Life Assurance Companies Act, 1870."

It would, then, appear that the gross premium valuation must be increased by some margin sufficient at least to cover the expenses for the future, but this will be far from the amount of the loading. It may be assumed that if such a fund is provided, in addition to the gross premium valuation, the Office will be in a solvent although not in a prosperous condition. In valuing the liabilities of the European, the writer of this paper adopted the terms of the 17 Offices' Experience mortality and 4 per cent. interest, with the addition of a fund equal to a loading of 10 per cent. of the aggregate gross premiums, as a measure of solvency. It will probably be admitted that an ordinary Office, with lives fairly selected, would be solvent on such a valuation, and even give profits if economically managed; and there can be no doubt but that if the assets of the Albert or

European had been equal to this standard the petitions against them must have been dismissed with costs. What should be the measure of solvency is a fair subject for discussion; but upon the insolvency of a Life Office the question in hand seems to tend to this simple form, whether the measure of solvency must not be also the just measure of proof. If, however, solvency can exist without the full reserve of a pure premium valuation, the adoption of such a valuation in a liquidation produces this remarkable result, namely, that by passing the Rubicon of insolvency we entirely alter the relative shares of claimants in the fund which is divisible. For example, in the European case the unpaid claims, on the 10th of June, 1871, amounted to £84,020—the gross premium valuation with the 10 per cent. charge was £1,311,143. Hence the share in the assets, whatever they turned out to be, to which the unpaid claimants on terminated policies were entitled, would be represented by the fraction $\frac{84020}{1395163}$, and they would be exactly paid if the assets amounted to £1,395,163, and we assumed this amount as the exact measure of solvency. But if this assumption is made, and the line must be drawn somewhere, and it is then found on re-examination that the assets are £100 deficient, the Company becomes insolvent; and if a pure premium valuation becomes imperative, the liability upon it is £1,652,844, to which must be added the unpaid claims, £84,020, as before. The share of the unpaid claimants in the assets is then represented by the fraction $\frac{84020}{1736864} \times (1395163-100)$; and by the loss of £100 that share is reduced to 80 per cent. of its former amount, and a deficiency created of £341,801.

There is, however, one view which appears so obvious that it is almost strange that it should have been passed unnoticed in Lancaster's case (p. 678). In discussing the mode in which annuities are to be valued, Lord Cairns observes:—

"I find with regard to annuities that in some of the Companies
" absorbed by the Albert valuations have already been made by the
" Court of Chancery, founded upon the tables of the Company
" granting the annuity. It is obviously a fair and proper mode of
" valuation, when you can find the tables of the Company which
" has granted an annuity, to take those tables as the basis of
" valuation. It is a right thing, if a person is to be reimbursed
" as it were the value of his annuity, that the value should be
" ascertained on the same scale on which the annuity was originally
" granted and paid for. The result therefore with regard to
" annuities is that in my opinion in every case a valuation of the

"annuity must be made as at the date of the winding-up order, "according to the tables of the Company originally granting the "annuity, and where those tables cannot be ascertained (which "possibly may be the case in some of the minor Companies), "then according to the table which, after consideration, I have "thought the best one to adopt—the table which is called the "Government Annuities Experience Table, in each case taking "four per cent. as the rate of interest."

Let us then apply the same reasoning to policies upon lives. If you take the tables of premiums of the Company which has granted the insurance, you here have a means of measuring the loss occasioned by the lapse of time which may be fairly accepted, and which may almost be said to have been approved by implication by both parties when the policy was issued. The difference between the two premiums at the age of the insolvency and at the age of the issue of the policy, when valued as an annuity by such a table as may most nearly represent the probable experience, would then give the value of the claim, but the table of premiums must be the non-profit scale, since all hope of profit is gone, and in profit policies the difference (if any) to be valued will be the difference between the non-profit premium at the age of valuation and the actual premium or profit premium at the age of entrance. The non-profit policy will of course be now more valuable than the profit policy, and so it should be; for both prove for the value of the sum assured, which is the same in both cases, and each submits to the deduction of the value of the premium, which is greater in the profit than in the non-profit policy. It is true that the profit policy may now be found to have no value, but this need not startle us, for we have seen that there is nothing unreasonable in the idea. It may happen that the Office has no non-profit scale, or that in an institution which is a congeries of defunct Companies like either of the two great insolvents, there is no common scale which can be reasonably adopted from the past practice. In such case a new scale must be taken according to the medium terms of other Companies of like pretensions, and this is nothing more than seeking the market price according to the practice of Insurance Offices—according to the suggestions in Bell's case—with this only difference, that in that case the Vice-Chancellor seems to have thought, with much reason, that if all could agree, a wholesale bargain could be made, when a benefit would be gained equivalent to a trade discount, such a difference as always exists between wholesale and retail transactions.

The following account of the discussion which followed the reading of the paper is abridged from the *Insurance Record.*

Mr. A. H. BAILEY believed the principle which Lord Cairns has enunciated to be sound, and would give the reasons for that portion of the decision. The problem is—an Insurance Company is avowedly insolvent and has stopped payment, the assets have to be realized, the expense of realization and distribution has to be deducted, and the net fund has to be divided amongst the several creditors in proportion to their *relative* interests. He laid emphasis on the word "relative," because this is something analogous to the rating of a parish, where the absolute value is of secondary importance and the relative value is everything. In order to arrive at the value of these you must ascertain the amount of damage sustained by each creditor. If you ask any ordinary policyholder what is the amount of damage he has sustained by the stoppage of the Company, the reply of nine out of ten would be, "The amount of premiums I have paid and lost is the amount of my damage." Any actuary would say, "No, that is a "mistake; you have had some return for the premiums paid, for you have "been assured for a term of years. The measure of your damage is the "difference between the premium for a whole life insurance and that for "the period during which you have been insured." Not to complicate the question, he would take the case of a man assured a year before the stoppage. The annual premium for the whole term of life is P_x, the annual premium for one year $(1-p_x)v$. So that the amount of damage sustained will be represented by the formula

$$\frac{\left\{P_x-(1-p_x)v\right\}\frac{1}{v}}{p_x}=\frac{P_x-v+vp_x}{vp_x}=1-\frac{v-P_x}{vp_x}=1-\frac{1+a_{x+1}}{1+a_x}$$

The first formula represents the amount of damage done, which is thus proved to be equal to the value of the policy. The same principles apply to any number of years. He thought it better to look at the case thus retrospectively and not prospectively, because in these insolvent Companies there is no future whatever. The question is not, what reserve has to be made to provide for the liability of the future, but what is the amount of damage occasioned by the misdeeds of the past? We have now to consider what is P_x; what is the premium P_x actually paid. And here he must join issue with Mr. Bunyon, and, to use his own forcible expression, say, that his views on the matter are "hopelessly misleading and erroneous." The premium on every assurance—not in any way from the manner in which it has been constructed, but from the nature of the case—consists of two parts—one, to provide for the risk, and the other the surplus or margin that is over. To divide that premium into these two component elements, is a problem of difficulty and delicacy, but it must be attempted. The cause of the damage to the unfortunate policyholders in the 55 Companies, forming the congeries of the two great insolvents, had been the ignorance of the managers as to what portion of the premiums might be devoted to other purposes than the risk; or, if not ignorance, then the recklessness, or worse, in applying that knowledge to the practical concerns in which they were engaged. And the damage done to the policyholders in consequence is exactly the extent to which the

proportion of the premium, whatever that may be, which should have been kept intact to meet the claims, has been trenched upon for other purposes. If we could arrive at the exact amount of that proportion, we could fix with extreme accuracy the amount of the damage in the case which Mr. Bunyon has put, as between the representatives of the claimants under policies which have lapsed and the existing assured. The object of Lord Cairns's judgment was to endeavour to carry out this principle with sufficient accuracy for practical purposes. He had heard with regret the statement made at a public meeting, that the insolvency of the European was due in a great measure to their not having been sufficiently careful in the selection of lives: that if they had rejected more lives, what has occurred would never have happened. He did not think any actuary would deny that the insolvency was due to trenching, in ignorance or recklessness, on the portion of the premiums that ought to have been reserved for future claims. To take in illustration the age of 40, where the mortality is almost exactly one per-cent. If 100 men at that age had insured their lives one year before the insolvency of the European was proclaimed, one out of the 100 would have died, and the contract of all the 100 would have been broken. It is admitted by all parties that the representatives of the deceased man must prove for the sum assured; and he believed that as between the representatives of the deceased and the 99 survivors, Lord Cairns's judgment maintains the relative proportion of the damage sustained by each quite accurately in principle, and with as much accuracy in practice as the case will allow. He had listened with attention to Mr. Bunyon's paper, but was unable to say what he proposes to substitute for these principles. It seemed to him, at the conclusion of the paper, that he proposed to substitute the decision of Lord Justice James in Bell's case for that of Lord Cairns in Lancaster's case. The decision in Bell's case is in effect to capitalize the difference between the premium at the age when the policy was effected and the present age. Now, the difference between that and the principle which Lord Cairns has laid down, would, as regards the relative interests of policy-holders, be so slight as not to be worth contending for. But Mr. Bunyon had, in a previous part of his paper, arrived at a conclusion which seemed to him to be wholly different. Mr. Bunyon maintained strongly that whatever be the measure of solvency must also be the measure of proof: and yet the very measure of solvency which he himself suggested in the case of the European is a wholly different measure from that which a valuation on the principles of the decision in Bell's case would give. And if he values by gross premiums, why take off 10 per-cent or any other percentage? Upon the amount of this deduction will hang the question whether a policyholder is a creditor or not. According to the principles Lord Cairns has propounded, every whole-term policyholder is to be a creditor. But let them fancy an appeal from the Vice-Chancellor to the Lords Justices, and then to the House of Lords, as to whether 9, or 10, or 11 per-cent shall be deducted, in order to determine whether or not a man is to prove at all. That the measure of solvency is to be the measure of proof, has at first sight something plausible in its favour, but he thought it would not bear investigation. When a Company is on its trial for solvency, and a petition presented to wind it up, it is something in the position of a man tried as a criminal, who, if there is any doubt, gets the benefit of the doubt.

A participating policyholder contracts for two things,—a promise to pay

the sum assured, and a speculation for future profits during the currency of the policy. But when the contract is broken, the only thing for which he can claim damage is the non-payment of the sum assured. The surplus is a speculation avowedly on both sides; and because the speculation has turned out to be a bad one, it was admitted most fully he cannot claim any damage on that account. At the same time, it seemed like adding insult to injury to say to such a man, "We cannot pay what was promised you, and also "your speculation has turned out badly, but because you have paid us more "than another man you are damaged less or not damaged at all." He could not follow that reasoning. With reference to Mr. Bunyon's remark as to the universal applicability of a pure premium valuation, he admitted fully that that method was not applicable in all cases; but he thought it applicable to the case of proof, when the insolvency of the Company has been admitted.

Mr. C. Walford thought the discrepancy between Lancaster's and Bell's cases arose, in a great measure, from the omission of a very important order of the Court, which is a legal interpretation of the 158th clause. The order (the 25th) is as follows:—"The value of such debts" (referring to the debts spoken of in clause 158) "and claims as are made admissible "to proof by the 158th section of the said Act shall, so far as is possible, "be estimated according to the value thereof at the date of the order to "wind up the Company." Then comes the simple question of the "value thereof." He believed with Mr. Bunyon that consequential damages cannot, in a legal point of view, enter into that value. It therefore narrows to that extent all those circumstances affecting the health of the party, or any circumstances arising incidentally out of the period for which the person has been assured. Whatever may have happened, short of death, since he effected the contract, does not actually reach the conditions of his contract, and therefore does not come into the matter of valuation from a legal point of view, inasmuch as, if it did, it would partake of the nature of consequential damage. Vice-Chancellor James, trying to find a rule which should admit of being placed alongside the instructions of the Act, hit upon two points, namely, the case of persons who had become in impaired health between the date of the insurance and the failure of the Office, and that of the with-profit policyholders. Lord Cairns, coming to a decision upon three points, and having before him the rule above quoted, together, no doubt, with the experience of Vice-Chancellor James in Bell's case, and with a far larger grasp and view of the whole case, laid it down that, substantially, the details which the Vice-Chancellor had tried to reach did not meet the merits of the case, and were not worthy to be estimated at all in forming part of the claims. He put it in this way, that whatever rule you lay down for proof does not very much matter, if you are all put upon the same basis, because there is only the pudding to divide, and if some get a larger slice than others, they must get it at the expense of others.

Mr. A. Hendriks believed the pure or net premium valuation was the proper process to be employed for an ordinary investigation of any Company's affairs, but did not go so far as to say that no other system is as good, inasmuch as there are other systems which produce about the same result. He thought also that when a Company is insolvent, the net method is not the one to apply to ascertain its liabilities. Mr. Bunyon had about hit the mark when he said that a 4 per-cent valuation and a reduction of 10 per-

cent from the premium would be about just and equitable. But for the degree of proof as a creditor he should still adopt the net plan. If he had a claim upon any estate, the assets not being equal to the amount which the creditors are entitled to, he did not go upon the amount which the estate will produce, but upon the amount which the estate ought to produce, and was secured according to the proportion of the assets for the amount of his original debt. Therefore, he thought that it reduces itself to the ordinary rule-of-three; the first term being the total amount which should be produced by the net premium valuation. This method gave the profit policyholder just the same sum as the non-profit policyholder; and he thought this was just. As to amalgamations, he thought, in most instances, they had been a benefit to the public. He thought the failure of the fifty-six Companies was due to the abuse of amalgamation. The majority of these Companies, instead of failing now, might have failed years ago but for amalgamating with other Companies. [Mr. PATTISON—It would have been much better if they had.] If these fifty-six Companies had been welded together, and if the old French proverb of *L'union fait la force*, had been carried out in its integrity, not only would they have existed at the present time, but they would have saved a vast amount to the public. If the various sticks which illustrate the proverb had been bound firmly and honestly together they would not have failed; but if you abstract from them the pith and leave only the bark bound together, putting the sticks and pith into the pockets of people who are not entitled to them, then it is impossible to carry out those benefits which would otherwise have accrued to the public in general. This illustration of amalgamation is borne out by many of the institutions which have honestly entered into amalgamations, and which have given, and will continue to give, very great benefit to the insuring public. It is difficult to say when a Life Insurance Company is insolvent; for if the European had fallen somewhat short of Mr. Bunyon's standard of a 4 per-cent valuation with a 10 per-cent margin, yet the business would have been taken over by the American Company, which would have assumed 5 per-cent interest and 10 per-cent margin, and not only would they have said that the European was solvent, but that it possessed a considerable surplus. The law is very defective upon that point, and naturally so, because it never contemplated that a Life Insurance Company would become insolvent.

Mr. PITCAIRN (of the Chancery Bar) thought it was difficult for a skilled actuary like Mr. Bailey to place himself in the position of an ordinary policyholder. For himself, not being an actuary, if he were a policyholder in an Insurance Company that had failed, he should consider that the measure of damages to him was the sum of money with which he could go to a solvent Insurance Company and buy a policy upon the same terms as the policy which he had lost. Lawyers are accustomed to put extreme instances, in order to show the merits or demerits of any proposition, and therefore he would say suppose that he, as a young man, had paid a low premium to an Office yesterday. In the meantime some accident has happened, and his life has become what is called uninsurable to-day. It may be, if Offices were a little more adventurous than they are, the true value of his life might be assessed, and if that were the case he should consider that, in his injured condition, his policy would be as valuable to him as if it were the policy of a very old man who had enjoyed good

health for a great number of years, and had paid in premiums perhaps more than the sum assured on his life. In such a case as that, he thought Lord Cairns's decision would work the highest injustice. He regretted, also, from another point of view, the difference of opinion that has taken place in the two decisions. The one we are told is law, and the other is not law. That may be very true in theory, but he thought that any lawyer, having to argue the same case over again in a court of law, would not be ignorant of the decision which has been given in the case of the Albert, and would bring that case to bear. The arguments of such a man as Lord Cairns cannot be passed over, and there will be great difficulty in saying which decision is to be followed. No doubt, if we are to follow the principle which Lord Justice James has laid down in Bell's case, and which, he thought, is just, there will be great difficulty in putting it into practice: there will be immense expense in proving in these cases of damage. But he believed the true, the proper, and just mode of winding up these Companies has not been applied. The proper mode, in his opinion, is to lump all the policies together and to prove for them; that is to say, that either by amalgamation, or by a temporary means of carrying on the business—because without the addition of new policies it would be impossible to continue the business altogether—[Mr. Bailey: "No"]—the proper way would be to issue new policies to the persons who have had the old ones reduced, and for them to pay the premiums reduced in proportion to the amount which they are to receive. Suppose that the Company has got into such a condition that it has only half the money necessary to get another Company to take over all its policy and annuity contracts. He would then hand them over, and arrange with the Office that took them that policies of only half the original amount should be issued, upon which only half the original premiums should be payable.

Mr. T. B. Sprague said the question came before them under two aspects—one was the legal aspect, the question of how the measure of proof is to be made according to Acts of Parliament, the statute law, and the regulations of the Court of Chancery in force for the time being; and the other was the measure of proof according to principles that actuaries think correct and just. He would not dwell at any length upon the first aspect of the question, but thought there was something to be said which had not yet been said about it. Mr. Bunyon had said that, as far as possible, the just value of each claim is to be estimated. Now, he (Mr. S.) thought that the method which Lord Cairns has laid down does not give the correct value to the claimant, "as far as is possible." Lord Cairns disregards the state of health of the claimant, which certainly throws out of consideration one most important element in determining the value of the claims. If a man has a policy at a certain premium, it is quite clear that, if he is in a bad state of health—if his expectation of life is less than the average—his policy is of much greater value than that of a person who is in robust health, who might go to another Office and get assured for a premium very little, if at all, exceeding that which he has been paying. Therefore, that decision does not carry out the principle of determining the value as far as possible. It might be difficult to lay down a scheme which should do so, but he did not think it impossible. One method he would suggest would be this—that all the persons having contracted with

the insolvent Company, not for a certain sum of ready money, but for a certain reversionary sum, if the values of their policies were estimated in reversion instead of in cash, that would be an approximation to determining the just value of their policies. A man has had a policy in force for so many years subject to certain premiums, then the value of that policy is now equal to that of a policy of a certain amount, upon which no future premiums are payable. Then you allot the dividend the estate allows of in reversion and not in cash. It is clear that that reversion will be of much more value to a man who is in bad health than to one who is in good health. The next step would be to devise some good plan by which the reversionary dividends of those who are in good health shall be bought up, and in doing that you must not take the ordinary average tables, but tables which shall exhibit a mortality much less than that of average lives; and if you value the policies of those who are in good health by such a table, you will have a large sum remaining behind which may be divided by agreement among those who remain, or the persons in impaired health. That is one way in which it might be done with approximate fairness. But he was still of opinion that it is impossible to arrive at a completely fair estimate of individual values by any process which can be adopted—that the only fair method of estimating the risks of an Insurance Company is to take them in a lump, and that it is impossible to do strict justice by attempting to fix a cash value upon each man's policy.

But leaving out of sight the question of health, and supposing all the policyholders to be in good health, he was entirely at issue with Mr. Bailey, and agreed almost entirely with the author of the paper. A man having an insurance of a certain amount, to which a certain bonus has been added, and which is subject to the payment of a certain premium, we must deduct the value of the premiums under proper deductions from the value of the sum assured and bonus; if it comes out that the value of the premiums so found is greater than that of the sum assured and bonus, then that man has clearly sustained no loss, his policy is worth nothing, and he has no right to claim at all. Put it in another way. This man has paid the full participating rate of premium to a shaky Office, and after three or four years the Office has stopped. What damage has he sustained? He can go to another Office, which will grant him a non-participating policy at a lower rate of premium than he has been paying. Clearly that man has sustained no loss, and his old policy is worth nothing. Or, to take another view. Supposing he tries to sell his policy; he will not be able to do so, because it is worth nothing. Therefore, from various arguments we come to the conclusion that a man may have paid premiums for a number of years to a Company, in respect of which he thinks he ought to be able to claim, though he really may have no claim, because the policy has no intrinsic value. You must not take a retrospective view of the policy, but a prospective view, and you must consider what the man is entitled to receive—the sum assured,—and what he is liable to pay in premiums; so that if he has got a policy that purported to give him a claim to the profits of the Company, and consequently made him liable to pay a higher premium, you must, in valuing his policy, take that higher premium into account and deduct its value, without making any allowance for the profits he hoped to get. Therefore, it is perfectly reasonable that those who have contracted for policies with profits should, *cæteris paribus*, have a smaller claim for

damages than those who have contracted for policies without profits. Notwithstanding what Mr. Bailey had said, he thought that the principle which Mr. Bunyon has put forward so prominently, and which he alluded to himself on a former occasion—that the measure of solvency should be the measure of proof—is the only correct and proper principle to go upon.

Mr. Bailey has spoken of the absurdity of having to appeal from one Court to another, and ultimately to the House of Lords, as to what percentage shall be taken off from the value of the gross premiums in estimating the values of the policies. There is nothing absurd in that. It is unfortunate such questions should arise, but probably they will arise before long, when questions as to the solvency of various existing Companies are argued. The only question of solvency which has ever yet come before the Court under the Life Assurance Companies Act, is that of the European, and then there was no question of taking off 9, 10, or 12 per-cent. Manifestly, whatever percentage had been taken off would have left the European insolvent. The Companies to which he referred would no longer be able to get rid of their liabilities, as was formerly the case, by a secret treaty with some Company in a more unsatisfactory position than themselves, in pursuance of which a large sum was paid to some negotiator, the transaction being in every way greatly to the disadvantage of the policyholders. Such were the amalgamations of the Albert and the European; and to them more than anything else was the calamitous downfall of those Companies owing. Now that process has, happily, been put an end to by the Life Assurance Companies Act. Any future amalgamation will, we hope, be exposed to the light of day. That unfortunately is a matter we cannot speak about with certainty, for even already one Company has retired from business since the passing of the Act—it is not known exactly upon what terms and conditions; but certainly the Act has done a great deal to put impediments in the way of these hole and corner amalgamations, and has thus taken the most effectual step towards preventing cases of disastrous ruin occurring in the future. He wished to express his satisfaction at the effect the Act is gradually producing—not only are accounts appearing from all quarters, including those of Companies whose transactions were hitherto wrapt in mystery, but he thought the Act would go far to remove the possibility, in the future, of their having to discuss such a question as that which had been brought before them this evening.

Mr. W. P. Pattison remarked that the difference between the decisions of Lord Justice James and Lord Cairns had been greatly exaggerated upon the main question. The decisions have, however, distinctly differed upon the question whether the health of the life assured should be taken into consideration in estimating the amount of his claim, and he was very much disposed to go with Mr. Bunyon in saying that it would be very desirable to consider the assured whose health has become impaired since his policy was effected, as having sustained greater damage than the man whose health continued good; but the difficulties of carrying out the scheme seemed quite insuperable. Lord Cairns has clearly pointed out those difficulties, and actuaries in their daily practice decline to allow an increased surrender value to the man whose health is impaired. In valuations, too, they take no account whatever of the health of the lives assured, even though knowing that some are at the point of death. With reference to the measure of proof, he fully admitted that the mode of estimating the position of an Office,

when its solvency or insolvency is to be determined, is entirely distinct from the system of valuation which should be adopted when the Office is a going and prosperous concern. In the former case the question is, can the Office fulfil its contracts? If it can, it is not to be wound up. A very small margin for expenses will enable the Office to carry on its business if it does not seek after new business; and that is the reason why the method adopted in the valuation of the European was a correct one, since if any Office had a sufficient sum of money to provide for the claims, allowing 10 per-cent for expenses, it is quite clear that a petition for a winding-up should be dismissed, as the Office would be able to pay 20*s*. in the £ to every policyholder. But it will be observed that the individual case does not arise at all up to the point before an Office is declared insolvent. Up to that time it is presumed that every contract will be fulfilled; but the moment an Office is declared insolvent the individual cases arise, and the principle of determining whether an Office is solvent or insolvent has nothing whatever to do with the individual damage sustained by the policyholders. The principle laid down by Lord Justice James was, he believed, this—a man has been insured for a certain number of years, say, in the Albert or European, and the Office stops payment. He is to go to an Office which charges the same rate of premium, and to ascertain what sum they will require down to give him a policy of the old amount and at the old rate of premium. That sum down is clearly the damage he has sustained, whether he has paid one, two, three, or twenty premiums. The principle laid down by Lord Cairns is an approximation to this rule, and is very convenient where the insolvent Office represents in itself a number of insolvent Offices, and where policies have been granted at twenty, thirty, or forty, different rates of premiums. The rule laid down by Lord Cairns is extremely simple in such a case; and as the object is to ascertain the *relative value* of the claims, it is a very convenient rule. As to amalgamations, it was only this morning he was looking carefully into the circumstances of one of the Companies transferred to the European. The Office had something like £100,000, and if its affairs had been correctly valued, it ought to have had £150,000 in hand to be in a sound condition. The Office was transferred to the European, and a sum of about £20,000 disappeared—so that only £80,000, or thereabouts, was handed over. It is quite true that not more than £20,000 was lost to the amalgamating Office, but if the Office had not been transferred, and it had been declared that only £100,000 was possessed, whereas £150,000 was required, the shareholders would have been good to pay even £200,000, or £300,000, if necessary. Therefore all the damage to the policyholders in that instance arose from the amalgamation. If the Company had continued a separate concern, with a separate share capital, the policyholders would have got 20*s*. in the pound instead of a small dividend.

Mr. Sprague had remarked that the question of deducting 10 per cent. from the gross premiums was a matter of little moment in the case of the European. That is quite true as applied to the present time, for the Office would, no doubt, have been found to be insolvent whatever assumption was made. But it was not true two or three years ago, when the first affidavits were sworn stating that the Company was insolvent. The answer made to those affidavits was that it was not right to take the net premiums, and that if the gross premiums had been taken there would have been found

a difference of some £750,000 between the liability stated in the affidavits made by actuaries in support of the petition, and that resulting from the valuation of the gross premiums, and upon that answer the European was allowed to go on for two years longer. Therefore, the percentage for expenses is the very essence of the question in the early stages of insolvency.

One other passage in the paper he wished to mention. Mr. Bunyon has adopted this 10 per-cent deduction, and thus thrown a number of policies out of the region of participation in dividend; and further than that, he has given the results of the European valuation after deducting 10 per-cent from the gross premiums, without allowance for negative values. Now, deducting 10 per-cent from the profit policies converts a number of them into assets; and that ought not to pass without protest, not even in the case of a single policy. The insolvency of Offices has been in a great part due to the capitalization of the loading, and the consequent conversion of new policies into assets, and that delusive system ought not to receive the slightest sanction.

Mr. Sprague explained that when he said that in the case of the European the question of percentage was of no consequence, he distinctly referred to the case which was tried before the Court. The first time the petition to wind-up came on for hearing, there was no means of stopping the business of the Company, even if it had been proved unable to meet its future liabilities. At the time that question did come before the Court, it was immaterial what percentage was taken off, because in any case the Company must have been admittedly insolvent.

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Mr. Ambrose Smith said that when policies of three or four years' standing are guaranteed a surrender value, they must have a value to the extent of that surrender value. It is hardly right to turn round and say, because the value of the premiums is in excess of the value of the sum assured, that there is no claim under that policy.

Mr. W. H. Manly said they had been told they should look entirely at the contract. Mr. Sprague had given the instance of a person going to a rotten Office, and paying a premium for two or three years, and had said that if then the Office breaks up, he has no claim whatever on the estate if another Office will insure him for the same amount on the non-profit scale for the same premium as he has been paying. Now, he (Mr. Manly) thought a better view of this doctrine would be got by taking something extraneous to insurance for an illustration. Take the case of a builder who has undertaken to build a house for a certain sum annually. After the party has paid two or three sums, and the contractor has dug a ditch and put in a few bricks, he fails to build the house; and his reply to any remonstrance is, "Well, if you go to another contractor, he will build the house for those "sums which I have left behind, and therefore you have no claim against "me." He thought that an analogous case to the one we are considering; and he did not suppose that any of the members would agree that the man, after he has paid certain sums and only got a few bricks in exchange, has no claim whatever against the contractor, because another contractor could possibly take it up without any remuneration from the first contractor. He thought there was one point in which Mr. Bunyon rather lost sight of the principle laid down by Lord Cairns in his judgment with reference to

annuities. Lord Cairns said, "It is obviously a fair and proper mode of "valuation, when you can find the tables of the Company which has "granted an annuity, to take those tables as the basis of valuation." The actual principle involved in that is that the Office was liable for such a sum as it would itself have required to grant an annuity upon the same terms as the existing one. When we apply that principle to the case of a policy, it comes to this—what single premium would the Office have required according to its tables, to grant a policy upon the existing terms? That either comes to the formula which Mr. Bunyon has given,

$$(P_{x+n}-P_x)(1+a_{x+n})$$

or, leaving out altogether the question of profit upon both sides, will eventually come to something like the formula which we get by a pure premium valuation, which shows Lord Cairns's judgment to be consistent.

Mr. E. H. GALSWORTHY considered the importance of Lord Cairns's decision as to annuities had hardly been felt to the full extent. In the case of one company which was transferred to the Albert, it was found that the annuities had been granted for about one-half or two-thirds the premium that they ought to have been; and the values put upon them under the principle laid down are just about one-half or two-thirds of what they would be if they were valued upon the principle of getting the price which another Office would require to grant them at. He was of opinion that in regard to the two insolvent Companies, the amalgamations had been one cause, but not the sole cause, of their failure—one of the most serious causes being their enormous expenditure. He thought that amalgamations on fair terms, properly carried out, were good things.

Mr. A. BADEN believed that amalgamations are not in themselves hurtful, but have only been so from the way in which they have been carried out, as part and parcel of a system which may best be described as reckless trading.

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Mr. MACFADYEN said that it was held by Mr. Bunyon that it was right, in calculating the amount that an assurer in an insolvent Office could prove for, to deduct from the value of the reversion the capitalized worth of the loaded premium, less expenses, thus placing the profit policyholder in a very much worse condition than the non-profit policyholder. On what principle did Mr. Bunyon stop at that point? His arguments ought to carry him further. If it be legitimate to press such part of the worth of the loading paid by a profit policyholder into the calculation as will make the policy value *nil*, why not, by using the whole of it, make the value negative, and thus, instead of allowing the insurer to prove against the Office, cause the Office to set up a claim on him? Mr. Bunyon's reasoning appeared logically to point to the conclusion that it would be right to recover from the assured the negative values of their policies. This seemed to him a *reductio ad absurdum* proof that Mr. Bunyon's arguments are fallacious.

Mr. W. WHITE believed that five-sixths of the policies effected in the country are on the with-profit scale; and he was glad, therefore, to find that to a considerable extent the voice of this meeting was that the value of with-profit policies is not really less than that of non-participating policies. If the principle of giving a less value to a profit policy than to

a non-profit policy under a winding-up were maintained, it would have a very injurious effect upon policyholders, and it would be difficult in future to get anyone to insure on the participating scale.

The PRESIDENT, in asking for a vote of thanks to Mr. Bunyon, observed that he still held the opinion he had uttered on a former occasion, that it is quite impossible to devise an equitable mode of winding up an insolvent Life Insurance Company.

In the first place, with regard to the question of health, Lord Cairns had decided that that question is not to be taken into consideration in valuing the liability of the grantors of the policy to the policyholders; but if a life happens to drop before the stoppage takes place, then the full amount of the policy is to be a claim. The result of that may be, in the case of a man dying the day before the stoppage takes place, his representatives would be able to prove to the full amount of the sum assured, while those of a man who died the day after would only prove for the value of his policy, estimated upon his being a healthy life. It had been pointed out by Lord Justice James that, as deterioration of life is one of the considerations which the Office undertakes to guarantee the policyholders against, no equitable adjustment could be made unless a proper appreciation of that deterioration were made in the valuation. The difficulty of making such a valuation was one reason that led him to the conclusion that you could never get an equitable plan for winding up in cases of insolvency. He thought that there was a great deal of truth in what Mr. Pitcairn had said as to the view the policyholder, who knew nothing whatever about pure or gross premiums, or loading, would take of the matter. He would say, "I have entered into a contract for the payment of a certain annual premium in consideration of a sum to be paid at my death to my family, but you tell me you are unable to complete your contract." If the Office were not insolvent, but chose to discontinue business, he would have a right to ask for such compensation as would enable him to go to another Office and get the same advantages as he had covenanted for with the first. But the failure of the Office has nothing to do with the amount of the policyholder's claim upon it. Of course, it makes the greatest difference as to what he can obtain from it, but his claim is the same, whether he gets one shilling in the pound or twenty, and we ought to consider the two questions entirely irrespective of each other. With regard to Lord Cairns's decision, there appeared to be a contradiction. He says, "How are the annuities to be valued? They are to be valued upon a scale at which the Office would have granted an annuity on a life of the advanced age." We suppose that the annuities are granted with the view of profit, so that, according to Lord Cairns, the annuity-holder is entitled to prove for the value of his annuity according to the Office tables, which are calculated to give a profit, while the policyholder is to have no such right. It must be remembered that the policyholders who have taken out profit policies have contributed a large portion of the fund which it is proposed to divide. They have contributed a much larger proportionate sum than the non-profit policyholders, and yet, as he understood the proposal in the present paper, they are to be entitled to less than those who have contributed a smaller proportion. This seemed an anomaly which can never be upheld on any principle of equity. Moreover, he wished to protest against the statement which he had heard over and over again, to

the effect that the man who takes out a profit policy enters into a speculation. He thought the system of profit policies one of the greatest securities for the community in the practice of life assurance, and most valuable. What was the contract entered into between the assured and the Office? It was this: "We say we will undertake for a certain "annual sum of money to pay another given sum of money at your "death. We make our estimates according to the best of our ability as "to the circumstances under which we shall accumulate the money we have "agreed to pay; that is to say with regard to the probable mortality and "rate of interest. The contract may last for 20, 30, 40, or 50 years. Those "circumstances which we rely upon may vary very greatly in the course "of that term, and, therefore, altho' we might be contented to take a "lower premium from you if we could be assured our estimate could be "certainly depended upon, yet to guard us against fluctuations we take a "higher rate of premium, and that higher rate of premium will, if our "calculations prove correct, give us a larger profit than we should be "content to accept if we could be quite secure of obtaining it. Supposing, "therefore, the profit to be realized, we shall be willing to return to you a "considerable proportion." That was not a system, perhaps, which could be defended on theoretical principles, but, nevertheless, it was a satisfactory system and one which contributed materially to the safety of Insurance Companies, and was, therefore, of advantage to the public. He did not think a person taking out a profit policy had entered into a speculation at all. The same thing is done with regard to other insurances. In the case of marine insurances, for instance,—in war time it was common to take a heavy premium for a ship bound from some port abroad, when neither the insurer nor the insured knew the precise circumstances under which the ship was to sail. It was a common practice to take a premium—say of ten guineas—with a stipulation that if the ship sailed with a convoy, and arrived safely, a portion of the premium should be returned. [Mr. Bailey. —That is the common practice now.] The person who goes in for a return of premium does not enter into a speculation. He pays the extra premium as a protecting premium which more than covers the risk, and if the risk turns out favourably, he gets a certain return. Whatever opinions, however, might be formed on the subject of this paper, it was very important that it should be fully criticized and discussed.

Mr. Bunyon, in reply, said that the discussion proved how difficult it is to get over preconceived notions. The profit policy is the favourite of the Insurance Offices, and, therefore, actuaries find it very difficult to mete out justice to anybody else who comes in contact with them. That appeared to be the foundation of the views enunciated on the other side. What really had been the contention all through was, that when a Company fails, a profit policyholder is to have a profit policy in some other Company. Now, if we look at it in that point of view, what can be more unreasonable? A profit policyholder is entitled to have, and can only have, his dividend, and it is unreasonable to propose that he is to be placed, notwithstanding the insolvency of the Company, in the same position as he would have been in if the Company had been in a condition to make a profit, by being given a profit policy in another Company. Now, he thought that very unreasonable, because you cannot alter the contract. The contract is simply that he shall have the sum assured and a pro-

portion of the profit that is made, and also that he shall pay a premium during the existence of the policy. Mr. Manly illustrated that question exactly with his statement about the builder. A builder has entered into a contract to build a house, and he is to receive certain sums of money for it. He digs the foundation and brings a few bricks on to the ground, and then fails. What measure of proof in bankruptcy can be brought against him? Why, the sum required to complete the house and nothing else. That is precisely our case, and shows that the insurer is entitled to prove for the amount of his contract, namely, the value of the sum assured, less the premiums—that is to say, if you put it the other way, the value of the money to build the house. [Mr. Bailey—The employer did not contract for a share in the builder's profits.] That would have placed him in a more difficult position, but here, if he contracted to share in the profits, he contracted for something which eventually has no existence whatever. Now, in spite of this buying something which turns out a nonentity, if you give him a pure premium valuation, you give him a value for that bonus. He had suggested the measure of value as the difference between the non-bonus premium on one side and the actual premium, which might be a bonus rate, on the other, because the man had absolutely lost that for which he speculated. Now, with regard to the general scope of this paper, he did not think he had laid down any rule too strongly. He had contended that a pure premium value is not admissible, and rather proceeded to a consideration of the principles upon which dividend should be actually made, because it is impossible for any actuary to say decisively what decision should be ultimately arrived at. That can only be arrived at by the Courts of Law. A suggestion was made that he had overlooked the general order of the Court of Chancery. Now that general order he did not overlook at all, but the reason he did not mention it was this—the general order is difficult to apply, and no doubt will be very shortly brought under the consideration of the Courts, and really it does not affect the principle which he has laid down, and, therefore, he thought it better to omit reference to it. The general order does not apply to Insurance Companies only, but to all proofs under the winding up of Companies, and it says, practically no doubt, that the date at which the winding up order is made shall be the date of proof, but that must be only so far as it is practicable. Now, take the case of the European, where the affairs have been held in abeyance for six months at least, and consider how exceedingly difficult it would be to apply the date of the winding up order as the date of the proof, remembering that by the Winding-up Act—that is to say the Act of 1862—which is conclusively binding, while the general order is not equally binding upon the Court, but only explanatory of and subject to the statute—that the Winding-up Act does say that the date of the winding-up shall commence from the filing of the petition. He did not think it necessary to go further into details, because those persons who are conversant with this question will see it raises many difficulties as to the date at which the valuation should be made, and that matter must ultimately be decided by the Court. That was the reason he did not allude to the general order. As to what he had said about the pure premium having no existence except in the mind of the actuary, what he meant was this: That the pure premium depends upon the rates of mortality and interest which are used.

There is no absolute pure premium. The pure premium has no independent existence in itself, but arises merely from the calculations which the actuary chooses to adopt. One gentleman observed that if the policy had a surrender value, then the policyholder had a right to claim, but that is not conclusive. If it can be shown a policy has no value, then it cannot be valued. [Mr. SPRAGUE—It may have a value by contract.] If you have a contract that must override everything else. He would illustrate that by this observation: Take the case of any Company which has been accustomed, from time immemorial, to use one method of valuation—take, for instance, the case of the Equitable. He could not conceive such a thing as the Equitable becoming insolvent, but it might be possible that the assets of the Equitable might be divided, and then it is perfectly clear that no better table or mode of valuation could be used than that table which has for the best part of a century been used by the Equitable—the Northampton Table. [Mr. SMITH.—" Many Offices advertize that after two or three years' premiums have been paid they will give surrender values."] They may for the sake of obtaining business do that, and they may be content to give something; but that does not prove that the policy is itself worth anything. In conclusion, he would ask their careful and candid consideration of the arguments adduced in the paper, which, to his mind, were perfectly conclusive; bearing in mind that if you are to use a pure premium valuation, you are going to entitle a policyholder to prove for those profits which he paid for but cannot obtain.

Observations on the Rate of Mortality in Infancy and Childhood.

By WILFRED ARTHUR BOWSER, *Actuary and Secretary of the London and County Provident Institution.*

[Read before the Institute, 26 February 1871.]

THE desirableness of obtaining trustworthy statistics as to the mortality prevailing in Infancy and Childhood has been frequently insisted upon by Actuaries; no apology therefore is needed for placing before the Members of the Institute the results of some original investigations into the subject. Whilst regretting that the materials at my disposal are not very extensive, I trust that this contribution to the subject of Vital Statistics will, nevertheless, be found to possess some degree of value and interest.

The difficulty of tracing a large number of individuals throughout the entire period between birth and death, in order to form an unexceptionable mortality table, may be said to be insuperable; and the only tables, with which I am acquainted, which trace a number of children from birth till death, or the date upon which the observations closed and the survivors were treated as still living, are those given by the late Rev. John Hodgson, M.A., in

his "Observations on the duration of life amongst the Clergy of England and Wales"; wherein he gives the statistics of 1087 Children born to Clergymen residing chiefly in the County of Kent; together with those of 1839 Children born to Parents residing in the parish of St. Peter in the Isle of Thanet.

The investigations which form the subject of the present paper originated partly from a desire to obtain further information upon the subject of infant mortality, but chiefly because, having occasion to value the engagements of an Annuity Society for Baptist Ministers, their Widows and Orphans,—a portion of whose obligations consisted of contingent temporary annuities to children,—I considered it advisable to ascertain the rate of mortality actually prevailing amongst the class of children with whom I had to deal. Accordingly in October 1870, I issued a schedule in the following form to each of the Members, requesting him, in a circular attached, to fill it up, giving the date of birth of *every* child born to him and the date of death of each one who had died. I pointed out the absolute necessity, for my purpose, that the greatest care should be taken to give the dates correctly; and it may be well to observe that in the prospect of an increase of benefits from the Society, in the event of the Valuation proving favourable as was anticipated, there was a strong inducement on the part of the Member to supply the desired information, and to give it accurately.

Boys.			Girls.		
Initials.	Born.	If Deceased, Date of Death.	Initials.	Born.	If Deceased, Date of Death.
	18	18		18	18
	18	18		18	18
	18	18		18	18
	18	18		18	18
	18	18		18	18
	18	18		18	18
	18	18		18	18
	18	18		18	18

With one exception only, the whole of the Members returned the schedule carefully filled up, the result being that 86 of them

were found to have had families, consisting of exactly 500 Children in all. This number however was insufficient from which to deduce any authoritative conclusions, so I had recourse to my connection with the Society to send a similar circular and schedule to all Baptist Ministers in England, Scotland, and Wales, of 10 years standing and upwards, requesting them, as a favour to the Society as well as to myself to furnish me with the statistics of their families. Wishing, if possible, to compare the mortality of English with Scottish Children, and in order to obtain a larger number of facts from Scotland than the Ministers of the Baptist denomination could supply, I addressed my circular to all the Ministers of the Independent denomination in that part of the Kingdom; but I had to abandon the attempt at making this interesting comparison, being unable to obtain the dates respecting more than 250 Children in Scotland altogether; suffice it to say that, as far as the statistics serve, they strengthen the opinion, held by many individuals, that Scottish children, especially the boys, are more robust than English. To all who did not reply within a fortnight I sent a reminder; and the final result, after no small amount of trouble and correspondence, was, that about three-fourths of the schedules issued were returned to me, giving particulars of 1153 additional children. With regard to the schedules that were not returned, I have good reason to believe that the Ministers to whom they were addressed were mostly unmarried, or if married, were without families; and I do not think the results are much, if at all, affected by the absence of returns from the few who were disinclined to give me the information; in proof of which opinion I may say that only three refusals were received, and that previous to combining the facts obtained from the Members of the Society, with those obtained elsewhere, I tested the trustworthiness of the latter, as compared with the former; and found, to my satisfaction, that the two sets of results were in sufficiently close agreement to entitle the Combined Experience to confidence. For instance, the mortality percent amongst the Children of Members of the Society was, during the first year of life, 10·843 for the Boys, and 8·517 for the Girls, as against 10·042 and 8·115 respectively by the Combined Experience. The process of taking out the ages of the children, either at death, or on the date of the return if then living, was performed by myself, and was subsequently verified by another person. Abstracts of the ages thus found were then made; the greatest care being employed in arranging the details, as well as in the subsequent computations, and the final results of

the enquiry are given in the statistical and monetary tables appended hereto. These results, then, exhibit the mortality which prevailed amongst 1653 Children, representing the families of 313 Protestant Dissenting Ministers, residing in all parts of England, Scotland and Wales. The observations are not continued after 30 years of age, under which age there are 22,502 years of life and 378 deaths tabulated.

One question of importance,—the social position of the parents of these children,—may with propriety be referred to here. Many of the Ministers from whom returns were obtained are in the receipt of very good, though not extravagant, incomes, most of them being in moderately comfortable circumstances. No doubt some are not so fortunately situated, but I am satisfied that these form but a small proportion of the whole; and I think it may be asserted, with considerable confidence, that the Ministers who have contributed the materials for these tables may, taken together, be regarded as middle class men. There is no reason to suppose that the health of their children has been in any way influenced by their professional avocation or social position; on the contrary, I believe that the probabilities of life deduced from their experience may be fairly considered to represent those of middle class children generally.

The relative number of the sexes born is that of 104·58 Males to 100 Females, which ratio is almost identical with that of the population generally, as shown in the Registrar-General's Reports (*vide* 31st Report, page xii). Having the date of birth of each one of the 1653 Children, I was able to separate the first-born children from those subsequently born, and a curious fact in connection with the excess of male over female births, and one, I believe, not hitherto noticed, seems to have been brought to light by so doing. Out of 314 *first-born* Children, 173 were Boys, and 141 were Girls, being in the proportion of 122 to 100; whereas out of the remaining 1339 *subsequently-born* Children the proportion of the sexes was nearly equal, viz., 672 Boys and 667 Girls. If, as might be inferred from these statistics, the excess of male births is almost entirely restricted to first-born children, the fact should, perhaps, be taken into consideration in cases of assurance against issue-male; unless the contingency is dependent upon the son living to the age of 21, in which event the much greater probability of the first-born child being a son is counter-balanced by the heavier mortality to which that son will probably be subject, as will be seen from the following figures:—

Mortality of first-born and subsequently-born Children during the first Year after Birth.

Order of Birth.	Boys.			Girls.			Both Sexes.		
	Subject to Mortality.	Deaths.	Mortality percent.	Subject to Mortality.	Deaths.	Mortality percent.	Subject to Mortality.	Deaths.	Mortality percent.
First born	172·	22	12·790	140·5	5	3·558	312·5	27	8·639
Subsequently born	664·5	62	9·330	660·5	60	9·084	1325·	122	9·207
Irrespective of order of birth	836·5	84	10·042	801·	65	8·115	1637·5	149	9·099

It will be noticed from the above Table that the mortality per-cent of the sexes during the first year after birth is, irrespective of the order of birth, 10·042 for the Boys, and 8·115 for the Girls. But if we exclude the *first-born* children of either sex, and refer to the mortality of those *subsequently born*, it will be seen that the difference, though still in favour of the female, is not nearly so great, being 9·330 for the boys and 9·084 for the girls, which proportion more nearly corresponds with the difference observable in the mortality of the sexes at subsequent ages, as shown in the next table. The inference is therefore drawn that the excessive mortality amongst males generally during the first year of life is chiefly due to the first-born boys; and to a very similar extent that the excess of male births is attributable to them. The very heavy mortality amongst first-born boys is worthy of notice, and is, perhaps, easily accounted for; but it is difficult to explain the remarkably small mortality amongst the first-born girls; however, as this anomaly occurs in the separate statistics both of the English and Scottish children, as well as in those of the children belonging to the Annuity Society, and those not so belonging, it is probable, notwithstanding the small number of observed facts, that we have here an indication of a law of nature, which, until now, has escaped observation, and which could only be thoroughly elucidated by more extensive investigations, similarly conducted.

As the numbers at each age are too limited to enable me to construct a mortality table for each sex separately, I have in the following table combined the figures in groups of quinquennial ages, keeping however ages 0, 1, and 2 to 4 distinct. The superior vitality of the female is clearly shown; the only groups in which the female appears to be subject to a heavier mortality than the male being at ages 2 to 4 and 25 to 29. In the first instance, the

difference is but small, although the peculiarity exists at the same ages in several other tables, and in the latter case the numbers subject to mortality are too few to warrant a conclusive opinion. In the case of both sexes the mortality rapidly decreases from birth till ages 10 to 14 when it reaches a minimum, after which it increases, the males presenting a maximum of mortality at ages 20 to 24, and a slightly diminished death rate at the succeeding quinquennium.

Mortality of Children of Dissenting Ministers.

Ages.	Boys.			Girls.			Both Sexes.			Ages.
	Subject to Mortality.	Deaths.	Mortality percent.	Subject to Mortality.	Deaths.	Mortality percent.	Subject to Mortality.	Deaths.	Mortality percent.	
0	836·5	84	10·042	801·	65	8·115	1637·5	149	9·099	0
1	733·	29	3·956	722·5	27	3·737	1455·5	56	3·847	1
2– 4	1978·5	28	1·415	1942·5	30	1·544	3921·	58	1·479	2– 4
5– 9	2793·	21	·752	2709·	20	·738	5502·	41	·745	5– 9
10–14	2088·5	11	·527	2006·5	10	·498	4095·	21	·513	10–14
15–19	1482·5	10	·674	1408·	9	·639	2890·5	19	·657	15–19
20–24	976·5	11	1·126	915·	9	·984	1892·	20	1·051	20–24
25–29	550·5	6	1·090	558·	8	1·434	1108·5	14	1·263	25–29
Total	11439·	200	..	11063·	178	..	22502·	378	..	Total

Reverting to the mortality during the first year of life, irrespective of sex and order of birth, I find that of children born alive.

1·005 percent died on the day of birth,
1·996 percent died within the first week of life,
2·843 percent died within the first month of life, and
4·487 percent died within three months after birth.

From this it would appear that the first three months of infant life are nearly as hazardous as the succeeding nine months, the mortality for the whole of the first year being 9·099 percent. With regard to the children dying on the same days as they were born, I believe that they were all born alive, inasmuch as their names or initials were given in each case.

Dr. Farr, in the Supplement to the Registrar-General's 25th Annual Report, refers to certain statistics which show that the mortality of infants, in France, during the years 1856-60, was within a fraction of 18 percent for the first year of life; that is, one-fifth more than the corresponding rate for England, and just double the rate indicated by the present observations. The deaths

during the first week were 2·912 percent of the births, as compared with 1·996 of the Ministers' Children, and during the first month they were 7·348 percent as compared with 2·843 percent.

In the English Life Table No. 3 Dr. Farr shows the mortality for each month of the first year of age, which enables us to make the following comparison. Out of a total of 100 deaths occurring during the first year,

Dr. Farr assigns	31·0 to the first month:	These statistics attribute		31·2
"	51·9 to the first 3 months:	"	"	49·3
"	75·6 to the first 6 months:	"	"	67·0
"	24·4 to the second 6 months:	"	"	32·0

The proportions for the first month we may regard as identical, and those for the first three months are very nearly the same, but the figures for the first 6 months are widely different. Dr. Farr makes the mortality for the first half of the birth-year three-fourths of that for the whole year, whereas the experience makes the proportion two-thirds only; so that whilst Dr. Farr's rate for the second half year is but one-third of the rate for the first half, the mortality of the Ministers' Children during the second six months was one-half of that during the first six months. In the case of the Ministers' Children I find that a marked increase of mortality took place amongst the Boys at the age of nine months, which increase, it has been suggested to me, is due to dentition. This explanation as to the increase of mortality at this particular period of infancy seems satisfactory, but to whatever cause it is due, an allowance should, I submit, be made in any table showing the mortality of each month of the first year; and I think it will be found upon close investigation that Dr. Farr has understated the proportionate mortality for the eighth, ninth and tenth months, more especially that for the ninth month.

The actual mortality percent for the four periods of six months each immediately succeeding birth are as follows:—

	Males.	Females.	Both Sexes.
0 Months to 6 Months......	5·947	6·215	6·078
6 Months to 12 Months......	4·345	2·017	3·203
12 Months to 18 Months......	2·965	1·788	2·382
18 Months to 24 Months......	0·984	1·977	1·480

From the above figures it is evident that any table professing to represent the mortality in the earliest years of life should be founded upon observations commencing at the birth of each child:

the introduction of children a few months old materially affecting the death rate.

In the following table the mortality percent for each year under 5 years of age, and the aggregate mortality under that age out of 100 children born, according to the present observations, are compared with that of various other Life Tables. It should be noticed, in reference to the mortality amongst Clergymen's Children, that the Rev. Mr. Hodgson's statistics embraced only 139 deaths under five years of age, also that the difference in the mortality of Males and Females at age 0 is very much greater than that observable in any other table. The percentages in the column headed "Isle of Thanet" are obtained from Mr. Hodgson's statistics of 1839 Children resident in that part of the County of Kent, referred to in the commencement of this paper.

A glance at the mortality of both sexes combined, in the comparative table, will show how unsuitable the Carlisle and English Life Tables are for the purpose of calculating the premiums for Children's Endowments, to which purpose they are frequently applied.

MORTALITY PERCENT.

1. *Males.*

Ages.	Sons of Dissenting Ministers.	Sons of Clergymen.	Isle of Thanet.	Peerage Families.	Healthy Districts (Farr).	English (3) Life Table.
0	10·041	11·287	11·751	7·821	11·280	16·359
1	3·956	2·469	5·078	1·625	3·508	6·429
2	1·746	1·709	2·005	·865	2·180	3·550
3	1·818	·877	1·146	·390	1·542	2·384
4	·633	·898	·647	·436	1·263	1·782
Aggregate	17·181	16·453	19·379	10·846	18·592	27·628

2. *Females.*

Ages.	Daughters of Dissenting Ministers.	Daughters of Clergymen.	Isle of Thanet.	Peerage Families.	Healthy Districts (Farr).	English (3) Life Table.
0	8·115	6·048	9·035	5·948	9·264	13·471
1	3·737	1·907	3·337	1·639	3·188	6·192
2	2·356	1·313	1·981	·705	2·171	3·539
3	1·395	1·131	1·351	·559	1·533	2·414
4	·808	·464	·826	·561	1·256	1·769
Aggregate	15·527	11·504	15·680	9·166	16·444	24·945

MORTALITY PERCENT—(*continued*).

3. *Both Sexes.*

Ages.	Children of Dissenting Ministers.	Children of Clergymen.	Isle of Thanet.	Peerage Families.	Healthy Districts (Farr).	English (3) Life Table.	Carlisle Table.
0	9·099	8·734	10·446	6·962	10·295	14·949	15·390
1	3·847	2·192	4·234	1·631	3·349	6·312	8·060
2	2·050	1·513	1·993	·791	2·175	3·544	6·492
3	1·609	1·002	1·246	·468	1·537	2·401	3·794
4	·720	·685	·734	·494	1·258	1·775	2·872
Aggregate	16·373	13·563	17·605	10·077	17·541	26·318	32·030

The Table of Mortality, deduced from these observations, has been based upon the combined Male and Female Experience, and the numbers living at ages 4 to 21 inclusive have been adjusted by means of Mr. Woolhouse's method, explained in the *Journal of the Institute* (vol. xv page 389), the numbers-living at ages 0, 1, 2 and 3 being left untouched. The values of single and annual premiums for Children's Endowments, payable at ages 14 and 21, at 3, 3½ and 4 percent interest, have been computed upon the basis of the adjusted table, and will, I trust, be found useful and trustworthy by those who may have occasion to employ them.

TABLE II.—*Adjusted Mortality.*

Age.	Adjusted.		$D_x = l_x \times v^x$		
	Number-living.	Decrement.	3 percent.	3½ percent.	4 percent.
0	100000	9099	100000	100000	100000
1	90901	3498	88253	87827	87405
2	87403	1791	82386	81591	80809
3	85612	1113	78347	77217	76109
4	84499	927	75076	73636	72230
5	83572	779	72090	70367	68690
6	82793	655	69338	67352	65432
7	82138	559	66786	64560	62418
8	81579	490	64399	61952	59609
9	81089	446	62148	59497	56972
10	80643	430	60006	57169	54479
11	80213	420	57947	54941	52105
12	79793	396	55965	52806	49838
13	79397	387	54065	50766	47684
14	79010	427	52235	48811	45626
15	78583	432	50439	46906	43634
16	78151	495	48701	45070	41725
17	77656	524	46983	43270	39867
18	77132	593	45307	41525	38074
19	76539	620	43649	39812	36329
20	75919	677	42034	38154	34648
21	75242	..	40446	36535	33019

Table I.—*Mortality amongst* 1653 *Children born to* 313 *Dissenting Ministers residing in Great Britain, embracing* 22502 *Years of Life and* 378 *Deaths, under* 30 *Years of Age.*

Age.	Boys. Entered the Age.	Boys. Existing.	Boys. Subject to Mortality.	Boys. Deaths.	Girls. Entered the Age.	Girls. Existing.	Girls. Subject to Mortality.	Girls. Deaths.	Both Sexes. Entered the Age.	Both Sexes. Existing.	Both Sexes. Subject to Mortality.	Both Sexes. Deaths.	Both Sexes. Mortality percent.	Both Sexes. p_x	Both Sexes. λp_x	Both Sexes. λl_x	Both Sexes. l_x	Both Sexes. d_x	Age.
0	845	17	836·5	84	808	14	801·	65	1653	31	1637·5	149	9·0991	·909009	·9585682	·0000000	100000	9099	0
1	744	22	733·	29	729	13	722·5	27	1473	35	1455·5	56	3·8475	·961525	·9829606	·9585682	90901	3498	1
2	693	12	687·	12	689	20	679·	16	1382	32	1366·	28	2·0498	·979502	·9910045	·9415288	87403	1791	2
3	669	18	660·	12	653	16	645·	9	1322	34	1305·	21	1·6092	·983908	·9929545	·9325333	85612	1378	3
4	639	15	631·5	4	628	19	618·5	5	1267	34	1250·	9	·7200	·992800	·9968618	·9254878	84234	607	4
5	620	13	613·5	7	604	16	596·	7	1224	29	1209·5	14	1·1575	·988425	·9949437	·9223486	83627	967	5
6	600	24	588·	7	581	26	568·	2	1181	50	1156·	9	·7785	·992215	·9966058	·9172923	82660	644	6
7	569	26	556·	1	553	19	543·5	3	1122	45	1099·5	4	·3638	·996362	·9984172	·9138981	82016	299	7
8	542	22	531·	2	531	29	516·5	4	1073	51	1047·5	6	·5728	·994272	·9975052	·9123153	81717	467	8
9	518	27	504·5	4	498	26	485·	4	1016	53	989·5	8	·8085	·991915	·9964745	·9098205	81250	657	9
10	487	23	475·5	4	468	26	455·	2	955	49	930·5	6	·6448	·993552	·9971906	·9062950	80593	520	10
11	460	33	443·5	2	440	28	426·	0	900	61	869·5	2	·2300	·997700	·9990000	·9034856	80073	184	11
12	425	24	413·	1	412	23	400·5	4	837	47	813·5	5	·6146	·993854	·9973226	·9024856	79889	491	12
13	400	20	390·	2	385	22	374·	2	785	42	764·	4	·5235	·994765	·9977205	·8998082	79398	416	13
14	378	23	366·5	2	361	20	351·	2	739	43	717·5	4	·5575	·994425	·9975621	·8975287	78982	442	14
15	353	24	341·	0	339	22	328·	3	692	46	669·	3	·4484	·995516	·9980482	·8950908	78[illegible]40	352	15
16	329	21	318·5	2	314	17	305·5	2	643	38	624·	4	·6410	·993590	·9972072	·8931390	78188	501	16
17	306	23	294·5	2	295	26	282·	1	601	49	576·5	3	·5204	·994796	·9977340	·8903462	77687	40[illegible]	17
18	281	13	274·5	3	268	19	258·5	3	549	32	533·	6	1·1257	·988743	·9950834	·8880802	77282	870	18
19	265	22	254·	3	246	24	234·	0	511	46	488·	3	·6148	·993852	·9973217	·8831636	76412	469	19
20	240	17	231·5	3	222	12	216·	2	462	29	447·5	5	1·1173	·988827	·9951203	·8804853	75943	849	20
21	220	21	209·5	1	208	15	200·5	2	428	36	410·	3	·7317	·992683	·9968106	·8756056	75094	549	21
22	198	9	193·5	3	191	18	182·	1	389	27	375·5	4	1·0652	·989348	·9953491	·8724162	74545	794	22
23	186	15	178·5	1	172	12	166·	2	358	27	344·5	3	·8708	·991292	·9962016	·8677653	73751	643	23
24	170	13	163·5	3	158	14	151·	2	328	27	314·5	5	1·5898	·984102	·9930401	·8639669	73108	1162	24
25	154	25	141·5	2	142	9	137·5	2	296	34	279·	4	1·4337	·985663	·9937284	·8570070	71946	1032	25
26	127	11	121·5	2	131	11	125·5	2	258	22	247·	4	1·6194	·983806	·9929094	·8507354	70914	1148	26
27	114	11	108·5	1	118	13	111·5	3	232	24	220·	4	1·8182	·981818	·9920310	·8436448	69766	1268	27
28	102	13	95·5	1	102	9	97·5	1	204	22	193·	2	1·0362	·989638	·9954764	·8356758	68498	710	28
29	88	9	83·5	0	92	12	86·	0	180	21	169·5	0	·0000	1·000000	1·0000000	·8311522	67788	0	29
30 and upwds.	79	79	..	..	80	80	..	..	159	159	..	..	..			·8311522	67788	..	30 and upwds.

Table III.—Children's Endowments.—*Nett Single Premiums for Endowment of* 1.

Age.	Payable at Age 14.			Payable at Age 21.		
	3 percent.	3½ percent.	4 percent.	3 percent.	3½ percent.	4 percent.
0	·52235	·48811	·45626	·40446	·36535	·33019
½	·56451	·52878	·49547	·43710	·39579	·35856
1	·59187	·55576	·52201	·45829	·41599	·37777
2	·63403	·59823	·56461	·49093	·44778	·40860
3	·66671	·63212	·59949	·51624	·47315	·43383
4	·69576	·66287	·63168	·53873	·49616	·45713
5	·72458	·69366	·66423	·56105	·51921	·48069
6	·75334	·72471	·69730	·58332	·54245	·50462
7	·78212	·75606	·73098	·60561	·56591	·52899
8	·81111	·78788	·76541	·62805	·58973	·55392
9	·84049	·82038	·80085	·65081	·61408	·57956
10	·87049	·85379	·83749	·67404	·63907	·60608
11	·90142	·88841	·87566	·69798	·66498	·63370
12	·93335	·92435	·91548	·72270	·69188	·66251
13	·96614	·96147	·95685	·73107	·71967	·69245
14	..	..	..	·77431	·74851	·72368
15	..	..	..	·80188	·77891	·75671
16	..	..	..	·83050	·81063	·79133
17	..	..	..	·86087	·84435	·82823
18	..	..	..	·89271	·87984	·86721
19	..	..	..	·92662	·91769	·90889
20	..	..	..	·96221	·95757	·95296

Table IV.—Children's Endowments.—*Nett Annual Premiums for Endowment of* 1.

Age.	Payable at Age 14.			Payable at Age 21.		
	3 percent.	3½ percent.	4 percent.	3 percent.	3½ percent.	4 percent.
0	·05293	·05086	·04881	·03073	·02892	·02718
½	·05617	·05401	·05187	·03172	·02988	·02818
1	·05890	·05677	·05465	·03326	·03141	·02969
2	·06541	·06324	·06105	·03586	·03397	·03214
3	·07294	·07071	·06845	·03869	·03676	·03488
4	·08189	·07962	·07551	·04182	·03986	·03793
5	·09282	·09049	·08804	·04534	·04334	·04137
6	·10646	·10406	·10150	·04932	·04729	·04526
7	·12398	·12151	·11910	·05388	·05180	·04980
8	·14734	·14478	·14228	·05914	·05702	·05497
9	·18004	·17738	·17476	·06529	·06313	·06105
10	·22912	·22631	·22354	·07257	·07036	·06822
11	·31096	·30794	·30494	·08132	·07907	·07687
12	·47473	·47128	·46786	·09205	·08974	·08748
13	·96614	·96147	·95685	·10549	·10311	·10079
14	..	..	..	·12281	·12036	·11796
15	..	..	..	·14595	·14342	·14094
16	..	..	..	·17843	·17579	·17320
17	..	..	..	·22724	·22447	·22172
18	..	..	..	·30877	·30576	·30279
19	..	..	..	·47204	·46861	·45245
20	..	..	..	·96221	·95757	·95296

On the Philosophy of Statistics. *By* W. S. B. Woolhouse, F.R.A.S.

[Reprinted from the "*Companion to the Almanac*" for the Year 1872.]

STATISTICS, generally considered, is a term of very comprehensive import, and is to be understood as having reference to an important collection of facts properly arranged and systematized in the form of numerical tables, for the purpose of conveying such information or data as may best assist in the investigation and discussion of particular subjects of inquiry. The general principles applicable to these investigations are, for the most part, intimately allied with the mathematical theory of probabilities, and constitute the true science of statistics. It will, moreover, be found, on examination, that the same identical principles lie at the foundation of all the physical and inductive sciences so far as they originally and necessarily depend upon experiment and observation.

The word statistics, derived from the Latin, *status,* signifies a state, condition, or standing. The subject of statistics has, however, by some writers, been unnecessarily restricted to that department of political science which is concerned in collecting and arranging facts illustrative of the condition and resources of a nation. It is chiefly to the advancement and growing importance of political economy that we are indebted for the cultivation of this particular branch of statistics. The fundamental doctrines of that science, which are directly concerned about the prosperity and happiness of mankind, were not reduced to any method or system until the middle of the last century; and since that time political economy has been more assiduously cultivated as an inductive science. The truth of preconceived theories has been practically tested by the observations and analysis of facts; and new principles have been gradually discovered and established by the same means. It therefore becomes the duty of a government to apply all the means in its power in aid of statistics, not only for general information and the administration of the affairs of state, but also for the advancement of political science.

In the report of the third meeting of the British Association it is stated that, in addition to the five then existing sections another, originating with some distinguished philosophers, had come into operation, the object of which was to promote *statistical* inquiries. The president, Professor Sedgwick, in justification of the addition of this sixth section, furthermore stated, that statistical inquiries may be made compatible with the objects of the British Association, "so far as they have to do with matters of fact, with

mere abstractions, and with numerical results. Considered in that light they give what may be called the raw material to political economy and political philosophy; and by their help the lasting foundations of those sciences may be perhaps ultimately laid." In the following year this new section was referred to as the prelude to the establishment of a flourishing society (the Statistical Society), which acknowledges itself the offspring of the Association, and which promises, by a procedure similar to that introduced by the Association, to advance materially the greatly neglected subject of British statistics.

From the original prospectus of the Statistical Society, printed at the end of the report of the third meeting of the British Association, we extract the following:—

"The Statistical Society of London has been established (15 March, 1834) for the purposes of procuring, arranging, and publishing 'Facts calculated to illustrate the condition and prospects of society.'

"The Statistical Society will consider it to be the first and most essential rule of its conduct to exclude carefully all *opinions* from its transactions and publications,—to confine its attention rigorously to facts, —and, as far as it may be found possible, to facts which can be stated numerically and arranged in tables.

" The first operation of the Society will probably be to subdivide and organise its General Council in such a manner as may enable that Council to deal conveniently with all the subdivisions of the subject-matter before it. Those subdivisions will necessarily be numerous.

"The whole subject was considered by the statistical section of the British Association at Cambridge, as admitting a division into four great classes:—

"1. Economical Statistics.
"2. Political Statistics.
"3. Medical Statistics.
"4. Moral and Intellectual Statistics.

"*Economical Statistics* comprehend, 1st, the statistics of the natural productions and the agriculture of nations; 2ndly, of manufactures; 3rdly, of commerce and currency; 4thly, of the distribution of wealth, or all facts relating to rent, wages, profits, &c.

"*Political Statistics* furnish three subdivisions: 1st, the facts relating to the elements of political institutions, the number of electors, jurors, &c.; 2ndly, legal statistics; 3rdly, the statistics of finance and of national expenditure, and of civil and military establishments.

"*Medical Statistics*, strictly so called, will require at least two subdivisions; and the great subject of population, although it might be classed elsewhere, yet touches medical statistics on so many points, that it would be placed most conveniently, perhaps, in this division, and would constitute a third subdivision.

"*Moral and Intellectual Statistics* comprehend, 1st, the statistics of literature; 2ndly, of education; 3rdly, of religious instruction and ecclesiastical institutions; 4thly, of crime. Although fourteen subdivisions have now been enumerated, it is probable that more will be required."

According to this restricted and stringent programme, to reason upon the facts presented, or to draw conclusions from them, is not within the province of statistics; but is the business of the statesman and of the political economist. And, in order still further to exemplify the precise character and limits of statistics, the emblem chosen for the Statistical Society consisted of a wheat-sheaf, with the motto "aliis exterendum." The statistical drudge was absolutely required to stifle his opinions; and to content himself with binding up his sheaves of wheat for others to thrash out!

It is satisfactory to know that, in the interests of science, these absurd restrictions have been necessarily disregarded in the numerous valuable papers that have appeared in the series of volumes of the Journal since issued by the Society. The statist usually sets out his results in numerical tables, because facts, especially when they exist in large numbers, are most compactly and clearly conveyed in tabular statements; and because he is not satisfied with giving deductions which may admit of question, but supplies the materials which every one may himself examine and compare. It is not true that statistics consist merely of columns of figures; it is required that all conclusions shall be fairly drawn from well-attested data, and shall be capable of mathematical demonstration. One of the principal objects of the statist is to ascertain the actual condition of his country and the causes of that condition, with a view to discover also the methods of improving it. The actual condition of the country is by no means obvious to the superficial observer, and the causes of the various phenomena which it exhibits lie very deep, and can only be elicited with the aid of extensive investigations conducted with skill and judgment as well as with mathematical exactitude. The statistics of education, of crime, of pauperism, of labour, of health, of trade, of agriculture, of manufactures, and of every one of the details which enter into the survey of our national condition and prospects are closely connected one with another. At the same time, not only do they admit of being studied separately, but more true progress will be made by such a method of study. In these inquiries the statist should know what information is wanted, and how he will look for it, before he commences his examination of a mass of figures; his mind should be unbiassed by any preconceived opinions and ready to receive the suggestions which the study is sure to produce. He should be ready at once to abandon any hypothesis as soon as he finds it to be untenable, and to subject his materials

to new tests as occasion arises. It will also be of great advantage that other minds should be brought into contact with his own, and that he should profit by the suggestions which their independent inquiries cannot fail to elicit. Statistics of population should have his special attention, as it is closely connected with so many other subjects.

Population is the statistical element of greatest importance: it relates, above all, to the people, and a just appreciation of their welfare and their requirements. It would be vain to attempt to form social statistics of value without having as a basis the results of a periodical census executed with every necessary precaution, and with all the accuracy which is so indispensable. Other data have no real value, except so far as they relate to the numbers of the population. The classification according to age supplies the means of establishing useful tables of population, of forming correct views on mortality, on the efficient resources at the disposal of the State in case of necessity, and of fixing the ratio between the useful portion which actively contributes to the general well-being and the weaker portion which yet requires assistance and support to become in its turn useful. The classification by professions, trade, and occupation, indicates the means by which the population provides for its subsistence, and allows the legislator more particularly to concentrate his attention on the separate details of the complicated machine confided to his care. The classifications by civil condition, by education, by origin, &c., furnish the administration with no less valuable information to assure internal good order, and to facilitate the execution of the laws. In fact, all questions which are connected with population deserve the greatest possible attention on the part of the Government.

Medical statistics are closely connected with the subject of population, and present the greatest difficulties to scientific investigation. The progress of medical science is necessarily slow and uncertain, for this reason, that the phenomena observed owe their origin to causes that are innumerable, and that, consequently, they seldom admit of being compared one with another. If there were a rigorous identity in all constitutions, then one disease followed by a cure, and well observed, would be sufficient to obtain a similar success every time that the same disease was reproduced in other individuals. But this perfect identity cannot reasonably be supposed to exist; and if it were absolutely requisite to foresee all the cases which could be presented, so as to

collect sufficient observations to verify all possible combinations, we should despair of ever arriving at anything satisfactory. Looking, however, at the broader lines of distinction, there does not really exist a very formidable number of essentially different cases, although the causes which influence one disease are in general both numerous and susceptible of variation. It is therefore only necessary to distinguish the more prominent causes which show a very decided individual influence, and to ascertain from a sufficient number of observations the degree of energy which belongs to each. In medical statistics relating to public health, in consequence of larger numbers, the facts are, in the aggregate, less complicated, and therefore more easily susceptible of a statistical analysis.

In the prosecution of medical science there are two distinct departments which may be conveniently separated. The successful cure of the patient depends on a compound event, viz.:—1, That the doctor shall discover the nature of the disease; 2, That he shall know the mode of treatment to be adopted. The former of these inquiries should have the precedence; diagnostics should always precede therapeutics. Investigation should first be directed to questions of diagnostics, or the distinguishing of symptoms. Such symptoms being given, what is the probability that the patient is affected by one malady rather than another? What is the probability of its duration, of its intermittence, and of its passing into a chronic state when it is abandoned to the operation of nature only? These questions, amongst others of equal importance, being once resolved, would constitute the surest basis for the study of the influence of medicine. In all these, as in other inquiries, the student should inform himself by observation, collect and analyse well-recorded facts, and render them as comparable as possible before seeking to discuss them with a view of deducing their relations, and methodically proceeding to the appreciation of causes.

The sciences of observation have for their object the study of the natural bodies by which we are surrounded and of the laws connected with them. They all proceed in a similar manner, and the same rules are generally applicable to them. In all of them it is equally found necessary to commence by collecting well-observed facts and classifying them systematically, so as to convey the required information. It is in weighing and appreciating the value and relative importance of the facts that science properly begins. The causes which might have produced them are sought,

their mode of action and their degree of energy studied, and an attempt is made to proceed from a knowledge of the past to a knowledge of what is to come. All sciences of observation at their commencement have undergone the same phases; they were formerly arts, for they were in the first stages necessarily limited to grouping, in a more or less successful manner, collections of observations and facts belonging to a particular order of things; and it is by the comparison and discussion of these facts that they afterwards became elevated to the rank of integral departments of science. If the subject of statistics presents itself in a similar manner and passes through the same phases, its future is not doubtful to those who can comprehend the sciences of observation in a philosophical point of view.

It should be remembered as a matter of general utility in the preparation and discussion of statistical tables, that material verifications, by means of differences, should extend to the columns of figures, as to the continuity of their progression. This object is singularly facilitated by the use of diagrams. A simple line enables us to appreciate at a glance a succession of numbers which the most subtle mind would find it difficult to retain and compare. The facility we thus have of at once perceiving a series of results, and of recognizing the peculiarities of their progression is such that we may at the same time follow another set of results, and judge by the common inflexion of the lines the ratios existing between the elements under comparison, and the common causes which may modify them simultaneously. Graphical representations cannot be too highly recommended when it is required to check series of numbers influenced by common causes, and to take a general view of the modes of action of such causes.

It has already been stated, at the commencement of this paper, that the scientific principles of statistics, properly considered, enter somewhat extensively into the calculus of probabilities. An account of this important connexion, exhibiting some of the higher mathematical applications, will perhaps comprise the most interesting and most scientific part of the subject now under consideration. Before entering upon this account it may be found useful to premise a short statement of the fundamental principles of the mathematical theory.

Probability is a numerical fraction expressing the value of the degree of presumption in favour of the occurrence of a particular event, when the causes or conditions which influence the result are partly known and partly indeterminate. The word chance has a

similar signification, and is more commonly used when the operating causes are so far unknown that there is almost a total absence of information, or data, to guide the mind to any settled conclusion, and when the individual result is not expected to approximate to any general average. In common phraseology the word chance means that we do not perceive any reason why the event should happen one way rather than another.

Every event is the result of some antecedent cause or combination of causes. A full knowledge of all the attending conditions and circumstances would be sufficient to determine the event. When, however, the several influences which combine in the production or non-production of an event are only partly known and partly indeterminate, the event is in such case indeterminate or contingent; it may or may not happen, so far as our limited knowledge extends, and the probability of its happening, or the presumption in favour of its occurrence, is to be ascertained by an accurate examination of the particular result under every possible combination of the indeterminate conditions.

In the theory of probabilities it must always be understood that the causes of events are considered only in reference to the number of chances they afford for the occurrence of those events which they may possibly, but do not necessarily, produce.

The simplest case of probability is that in which the event can happen in a certain number of ways, and may fail in another given number of ways, and in which the circumstances are such, that amongst these several ways all are equally likely to happen. The probability of the occurrence of the event is in such case accurately measured by the fraction of which the numerator is the number of possible ways in which the event can happen, and the denominator the total number of ways. That is—

$$\text{Probability of event} = \frac{\text{Number of ways it can happen.}}{\text{Total number of ways.}}$$

In the application of this rule it should be carefully remembered that the implied condition that the several ways are equally likely to happen is absolutely essential to its accuracy. For this it will be sufficient that no reason exists why one way should happen rather than another. Mistakes have frequently been fallen into through want of attention to this necessary condition.

We shall proceed with a few numerical examples on this and some other most important cases. It will be understood by the reader that the limits of this article will not admit of giving the mathematical investigation of the principles, but the several relations will always be distinctly stated and an example worked out at length.

CASE I.—Suppose that a bag contains 3 white balls and 5 black balls, and determine the probability of drawing therefrom a white ball in a single trial.

Here each of the 3 white balls may possibly be the ball withdrawn, which gives 3 ways in which the event can happen; whereas the total number of ways is 8, there being one for every separate ball, and it is obvious that each of these ways is equally likely. Therefore the required probability of drawing a white ball is $\frac{3}{8}$; and similarly the probability of drawing a black ball is $\frac{5}{8}$.

The true signification of these results may be thus interpreted. If a large number of similar trials be made, a white ball may be expected, on an average, to be drawn at the rate of 3 times in every 8 trials; and a black ball at the rate of 5 times in every 8 trials.

The sum of the two probabilities is $\frac{3}{8}+\frac{5}{8}=1$, and this must evidently represent the aggregate probability of drawing *either* a white or a black ball; and as this total event is always certain of fulfilment, it follows that in evaluating probabilities *unity* always represents *certainty*, and is the highest order of probability.

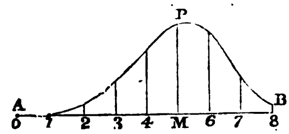
The foregoing case is perfectly analogous and in all respects equivalent to the following. A regular octahedron has 3 of its triangular faces of a white colour and the 5 remaining faces are black. If it be thrown indiscriminately, the probability that the uppermost face shall be white is $\frac{3}{8}$, and the probability that it shall be black is $\frac{5}{8}$.

CASE II.—A regular octahedron has 3 white triangular faces and the 5 remaining faces are black. It is required to find the respective values of the probabilities that in 8 throws a black face shall be presented uppermost 0, 1, 2, ... 8 times.

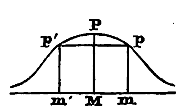
Adopt the fixed notation $\alpha=3$, $\beta=5$; then if $p=\frac{\alpha}{\alpha+\beta}$, $q=\frac{\beta}{\alpha+\beta}$ denote the initial probabilities of white and black for a single trial, the required probabilities will be the values of the respective terms of the expansion of the binomial $(p+q)^{\alpha+\beta}$, viz.: $p^8+8p^7q+28p^6q^2+56p^5q^3+70p^4q^4+56p^3q^5+28p^2q^6+8pq^7+q^8$, and the indices of p and q in each term indicate the respective numbers of white and black faces. Hence, employing the above values of p and q, viz., $\frac{3}{8}$ and $\frac{5}{8}$, we thus obtain, to four places,

Probability of	8	white and	0	black	. .	·0004
„	7	„	1	„	. .	·0052
„	6	„	2	„	. .	·0304
„	5	„	3	„	. .	·1014
„	4	„	4	„	. .	·2112
„	3	„	5	„	. .	·2816
„	2	„	6	„	. .	·2347
„	1	„	7	„	. .	·1118
„	0	„	8	„	. .	·0233
						1·0000

By way of illustrating the particular distribution of the several events the values of the respective probabilities are laid off on equidistant ordinates in the annexed diagram, and a continuous curve line is traced through the resulting points. This curve is remarkably regular in its form when we consider the smallness of the numbers that enter into the calculation. It is at once perceived that it exhibits the chief peculiarities of the species of curve which represents the facility of error; with which it, in fact, becomes identical when large numbers enter, and we make $\alpha=\beta$ in order to equalize the probabilities of positive and negative errors. The most probable event, viz., 3 white and 5 black, which appertains to the maximum ordinate PM, is that which theoretically presents the true average of an indefinite number of trials, and the segments of the abscissa measured from M, positively and negatively, represent the departures from the general average, and may be regarded as the errors in respect of the other events.



Errors of observation are of a similar description. A numerical result derived from observation must be expected to involve an error, the magnitude of which is dependent upon the character of the inquiry and the circumstances attending it. Each value must be considered as probably greater or less than the truth: in the former case it is said to be affected with a positive, in the latter with a negative error. By an error, it must be understood that we include only those accidental deviations of which no account can be given, except that they are found to exist. No variation is to be called an error which is more likely to be of one sign than the other. Errors beyond a certain magnitude should evidently be regarded and estimated as extremely improbable, and positive and negative errors should be regarded as equally probable. The curve which indicates the law of the facility of error



must then be of the kind shown in the annexed diagram, where M represents the position of the mean or most probable value; Mm a positive error; Mm' a negative error; and each ordinate of the curve exhibits the relative probability of the corresponding error. The probability of any accurately-defined value must be inappreciable, since it is only one out of an infinite number of possible cases; but the probability of the error falling between two stated values will be finite and will be properly represented by the curvilinear area comprised between the corresponding ordinates. As the error increases on either side of the mean value the probability or facility appertaining to the error must obviously diminish rapidly in magnitude, which circumstance is indicated at each extremity by the asymptotic approach of the curve towards the line of abscissæ. If x denote Mm, the magnitude of the error or variation from the most probable value, the standard law of facility in regard to large numbers is—

$$y = \sqrt{\frac{c}{\pi}}\, e^{-cx^2} = \mathrm{Y} e^{-cx^2}$$

where y denotes the ordinate pm, and Y the maximum ordinate PM. The indeterminateness of the constant c will allow of an adjustment of the law, with a considerable degree of nearness, to any law which may be presented; and its assumption will, in fact, give a near mathematical approximation to any other law, and the more nearly, the greater the number of observations.

The constant c is called the modulus of the law of facility, and its value determines the general amplitude of the variations from the mean value. As a general rule, the range or amplitude of divergency varies inversely, and the general persistency of the results varies directly, as the square root of the modulus of facility. The average divergence is $\frac{1}{\sqrt{\pi c}}$.

On account of the last-mentioned properties the modulus of facility of error (c) is also technically called the *weight* of the observations. For a given set of individual observations its numerical value may be determined as being equal to one-half the reciprocal of the average amongst the squares of the several estimated errors. The probable error of a single observation is found by dividing ·4769* by $\sqrt{c}$; and the probable error of the mean of n observations is found by dividing ·4769 by $\sqrt{nc}$.

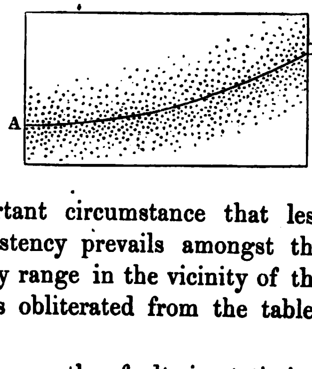
The mathematical theory of errors of observation is especially

* ·4769 is the particular value of t, which, in the table on p. 50, gives $\mathrm{P} = \frac{1}{2}$.

applicable to statistical tables which exhibit masses of observed facts properly classified, and, when the observations are numerous, there is a similar law of facility with respect to the variations of the individual facts from the mean values. In either case the deviations from the mean values are due to casual and indiscriminate circumstances which indifferently augment or diminish the several values; it being presumed that all disturbing conditions, which would have any assignable influence upon the results, have been duly eliminated in forming the tables.

A useful practical test of the reliance that should be placed in an aggregate table may sometimes be obtained by dividing the whole of the values observed into two or more portions, and ascertaining the relative proportions existing amongst the separate items, and noting how much they differ one from the other.

The importance of these considerations will be clearly appreciated on an inspection of the accompanying diagrams. The masses of points, in each, are supposed to represent a number of separate observations, and the central line AB is supposed to represent the average result after they are collected in mass and embodied in a statistical table. It is obvious that the former table will have more signification and will possess much greater value than the latter in consequence of the important circumstance that less divergency exists and greater persistency prevails amongst the separate observations, which generally range in the vicinity of the mean values. This fact, however, is obliterated from the tables when they exhibit the totals only.



Besides defects of this kind, there are other faults in statistical methods of inquiry, which it may be useful to notice, viz.:—

1. Attributing results to conditions or causes which are only partially or defectively stated.

2. Assuming certain elementary conditions to be independent and equally probable which are not so.

3. Comparing quantities which do not admit of satisfactory comparison, the same not having been observed under similar or corresponding circumstances.

4. Reasoning upon insufficient numbers, which cannot be free from the disturbing effects of accidental causes. It is, unfortunately, a too common failing to draw precise, formal, and confident deductions from a too limited number of observed facts.

It is almost needless to add that these and other imperfections will disappear when the fundamental principles are more generally understood and the science of statistics becomes more extensively investigated.

The use of the following table (page 50), showing the values of an important integral, will appear in the cases which now follow.

Case III.—In 6,000 throws with a die, what is the chance that the number of aces shall fall within the limits $1{,}000 \pm 50$; that is, shall lie between 950 and 1,050, both inclusive?

The particular event in which the numbers are proportional to the chances is the most probable result; and this is $\alpha = 1000$, $\beta = 5000$, where α denotes the number of throws which are aces, and β the number which are not aces. The proposed limits being $\alpha \pm x$, the formulæ for calculation are:—

$$c = \frac{\alpha+\beta}{2\alpha\beta}; \quad t = (x + \tfrac{1}{2})\sqrt{c}; \quad \text{required probability} = P_t,$$

this last value being taken out, with the argument t, from the table on page 50.

Five figure logarithms are convenient for these calculations; and, for the present case, the following is the work:—

$\alpha+\beta=$	6,000	log 3·77815
$2\alpha\beta=$	10,000,000	log 7·00000 sub.
		$\log c$ 6·77815
		$\log\sqrt{c}$ 8·38907
$x+\frac{1}{2}=$	50·5	log 1·70329
$t=$	1·2370	$\log t$ 0·09236
P=	·9198, the required probability.	

The chance therefore exceeds 9 to 1 in favour of the aces falling within the limits mentioned.

The following interesting example is given by Laplace:—

Example 2.—Supposing 18 boys are born to 17 girls, in which case out of 14,000 births the most likely individual result is, that 7,200 should be boys, and 6,800 girls; what is the probability that the number of boys shall fall between $7{,}200 \pm 163$ inclusive?

Calculation.

$\alpha+\beta =$	14,000	log 4·14613
$2\alpha\beta =$	97,920,000	log 7·99087 sub.
		log c 6·15526
		log $\sqrt{c}$ 8·07763
$x+\frac{1}{2}=$	163·5	log 2·21352
$t=$	1·9550	log t 0·29115
$P=$	·9943	

The odds are therefore nearly 175 to 1 in favour of the event.

Case IV.—In 12,000 throws of a die, what are the respective chances?—1, that the number of aces shall be exactly 2,000; 2, that the number of aces shall fall within the limits 1,970 and 1,999; and 3, that the number shall fall within the limits 2,001 and 2,030, both inclusive in each case.

Here $\alpha=2{,}000$, $\beta=10{,}000$, $x=30$; and the formulæ for calculation are

$$c=\frac{\alpha+\beta}{2\alpha\beta};\ Y=\sqrt{\frac{c}{\pi}};\ t=x\sqrt{c};\ y=Ye^{-t^2};$$

1. Chance that the aces shall be $\alpha\ =Y$
2. Ditto with limits $\alpha-x$ and $\alpha-1=\frac{1}{2}P-\frac{\alpha}{\alpha+\beta}(Y-y)$
3. Ditto „ „ $\alpha+1$ „ $\alpha+x=\frac{1}{2}P-\frac{\beta}{\alpha+\beta}(Y-y)$

where, as before, P is to be taken from the table on page 50.

Calculation.

$\alpha+\beta$	12,000	log 4·07918	log $\sqrt{c}$	8·23856
$2\alpha\beta$	40,000,000	log 7·60206 sub.	$x=30$	1·47712
		log c 6·47712	log t	9·71568
		log $\sqrt{c}$ 8·23856	2 log t	9·43136
		log $\sqrt{\pi}$ 0·24857 sub.	const.	9·63778 (log ^2e)
		log Y 7·98999	log	9·06914
	Prob. (1) Y=	·0098		0·11726 sub.
	$y=$	·0075	log Y	7·98999
	$Y-y$	·0023	log y	7·87273

t	P	Diff.	t	P	Diff.	t	P	Diff.	t	P	Diff.
·00	·0000	113	·64	·6346	74	1·28	·9297	22	1·92	·9934	3
·01	·0113	113	·65	·6420	74	1·29	·9319	21	1·93	·9937	2
·02	·0226	112	·66	·6494	72	1·30	·9340	21	1·94	·9939	3
·03	·0338	113	·67	·6566	72	1·31	·9361	20	1·95	·9942	2
·04	·0451	113	·68	·6638	70	1·32	·9381	19	1·96	·9944	3
·05	·0564	112	·69	·6708	70	1·33	·9400	19	1·97	·9947	2
·06	·0676	113	·70	·6778	69	1·34	·9419	19	1·98	·9949	2
·07	·0789	112	·71	·6847	67	1·35	·9438	18	1·99	·9951	2
·08	·0901	112	·72	·6914	67	1·36	·9456	17	2·00	·9953	2
·09	·1013	112	·73	·6981	66	1·37	·9473	17	2·01	·9955	2
·10	·1125	111	·74	·7047	65	1·38	·9490	17	2·02	·9957	2
·11	·1236	112	·75	·7112	63	1·39	·9507	16	2·03	·9959	2
·12	·1348	111	·76	·7175	63	1·40	·9523	16	2·04	·9961	2
·13	·1459	110	·77	·7238	62	1·41	·9539	15	2·05	·9963	1
·14	·1569	111	·78	·7300	61	1·42	·9554	15	2·06	·9964	2
·15	·1680	110	·79	·7361	60	1·43	·9569	14	2·07	·9966	1
·16	·1790	110	·80	·7421	59	1·44	·9583	14	2·08	·9967	2
·17	·1900	109	·81	·7480	58	1·45	·9597	14	2·09	·9969	1
·18	·2009	109	·82	·7538	57	1·46	·9611	13	2·10	·9970	2
·19	·2118	109	·83	·7595	56	1·47	·9624	13	2·11	·9972	1
·20	·2227	108	·84	·7651	56	1·48	·9637	12	2·12	·9973	1
·21	·2335	108	·85	·7707	54	1·49	·9649	12	2·13	·9974	1
·22	·2443	107	·86	·7761	53	1·50	·9661	12	2·14	·9975	1
·23	·2550	107	·87	·7814	53	1·51	·9673	11	2·15	·9976	1
·24	·2657	106	·88	·7867	51	1·52	·9684	11	2·16	·9977	2
·25	·2763	106	·89	·7918	51	1·53	·9695	11	2·17	·9979	1
·26	·2869	105	·90	·7969	50	1·54	·9706	10	2·18	·9980	0
·27	·2974	105	·91	·8019	49	1·55	·9716	10	2·19	·9980	1
·28	·3079	104	·92	·8068	48	1·56	·9726	10	2·20	·9981	1
·29	·3183	103	·93	·8116	47	1·57	·9736	9	2·21	·9982	1
·30	·3286	103	·94	·8163	46	1·58	·9745	10	2·22	·9983	1
·31	·3389	102	·95	·8209	45	1·59	·9755	8	2·23	·9984	1
·32	·3491	102	·96	·8254	45	1·60	·9763	9	2·24	·9985	0
·33	·3593	101	·97	·8299	43	1·61	·9772	8	2·25	·9985	1
·34	·3694	100	·98	·8342	43	1·62	·9780	8	2·26	·9986	1
·35	·3794	99	·99	·8385	42	1·63	·9788	8	2·27	·9987	0
·36	·3893	99	1·00	·8427	41	1·64	·9796	8	2·28	·9987	1
·37	·3992	98	1·01	·8468	40	1·65	·9804	7	2·29	·9988	1
·38	·4090	97	1·02	·8508	40	1·66	·9811	7	2·30	·9989	0
·39	·4187	97	1·03	·8548	38	1·67	·9818	7	2·31	·9989	1
·40	·4284	96	1·04	·8586	38	1·68	·9825	7	2·32	·9990	0
·41	·4380	95	1·05	·8624	37	1·69	·9832	6	2·33	·9990	1
·42	·4475	94	1·06	·8661	37	1·70	·9838	6	2·34	·9991	0
·43	·4569	93	1·07	·8698	35	1·71	·9844	6	2·35	·9991	1
·44	·4662	93	1·08	·8733	35	1·72	·9850	6	2·36	·9992	0
·45	·4755	92	1·09	·8768	34	1·73	·9856	5	2·37	·9992	0
·46	·4847	90	1·10	·8802	33	1·74	·9861	6	2·38	·9992	1
·47	·4937	90	1·11	·8835	33	1·75	·9867	5	2·39	·9993	0
·48	·5027	90	1·12	·8868	32	1·76	·9872	5	2·40	·9993	0
·49	·5117	88	1·13	·8900	31	1·77	·9877	5	2·41	·9993	1
·50	·5205	87	1·14	·8931	30	1·78	·9882	4	2·42	·9994	0
·51	·5292	87	1·15	·8961	30	1·79	·9886	5	2·43	·9994	0
·52	·5379	86	1·16	·8991	29	1·80	·9891	4	2·44	·9994	1
·53	·5465	84	1·17	·9020	28	1·81	·9895	4	2·45	·9995	0
·54	·5549	84	1·18	·9048	28	1·82	·9899	4	2·46	·9995	0
·55	·5633	83	1·19	·9076	27	1·83	·9903	4	2·47	·9995	0
·56	·5716	82	1·20	·9103	27	1·84	·9907	4	2·48	·9995	1
·57	·5798	81	1·21	·9130	25	1·85	·9911	4	2·49	·9996	0
·58	·5879	80	1·22	·9155	26	1·86	·9915	3	2·50	·9996	0
·59	·5959	80	1·23	·9181	24	1·87	·9918	4	2·51	·9996	0
·60	·6039	78	1·24	·9205	24	1·88	·9922	3	2·52	·9996	1
·61	·6117	77	1·25	·9229	23	1·89	·9925	3	2·53	·9997	0
·62	·6194	76	1·26	·9252	23	1·90	·9928	3	2·54	·9997	0
·63	·6270	76	1·27	·9275	22	1·91	·9931	3	2·55	·9997	0

$$\begin{cases} t=0\cdot5196 \\ P=\ \cdot5376 \end{cases}$$

$\frac{1}{2}$P	·2688	$\frac{1}{2}$P	·2688
$-\frac{1}{6}(Y-y)$	-4	$-\frac{5}{6}(Y-y)$	-19
Prob. (2)=	·2684	Prob. (3)=	·2669

Collecting these results we have

1.	Probability of exactly	2000 aces=	·0098
2.	„	1970 .... 1999 aces=	·2684
3.	„	2001 .... 2030 aces=	·2669
	Total, probability of	2000 ± 30 aces=	·5451

The last mentioned total, if alone required, is more readily found by the method of Case III.

The foregoing cases serve to calculate the probabilities of specific events from a knowledge of the circumstances under which they are to happen or fail. These are direct questions of probability. In Case V, which follows, we enter on inverse questions of probability; from a knowledge of the event which has taken place it is required to find the probability which results therefrom with reference to a particular set of circumstances under which it might have happened. These calculations are founded upon the general principle that when an event may have happened in several different ways, that way which is most likely to bring about the event is the most likely to have been the cause.

CASE V.—Supposing that in 1,000 trials an event has occurred 600 times and failed 400 times, what is the probability that the true chance of the occurrence of the event in one trial is comprised within the limits $\frac{600 \pm 30}{1000}$ inclusive?

Here $\alpha=600$, $\beta=400$, $x=30$;

$$c=\frac{\alpha+\beta}{2\alpha\beta};\ t=x\sqrt{c};\ P_t=\text{the required probability.}$$

Calculation.

$\alpha+\beta$	1,000,	log 3·00000
$2\alpha\beta$	480,000,	log 5·68124 sub.
		log c 7·31876
		log $\sqrt{c}$ 8·65938
	$x=30$	log 1·47712
$\begin{cases} t=1\cdot3693 \\ P=\ \cdot9472 \end{cases}$		log t 0·13650

If minute accuracy be required the result might be multiplied by $1+\frac{c}{6}-\frac{13}{12(\alpha+\beta)}$ giving ·9541 for the corrected probability. The odds are therefore more than 19 to 1 in favour of the proposed supposition.

The next case will enable us to judge of the probability of future events from those which are past, on the supposition that the circumstances under which the latter were produced continue in operation. This is essentially the very problem, above all others, that must be continually presented in the course of practical experience. It is mathematically discussed by referring the event to every possible cause, deducing the probability of each separate cause from the event, and then deducing the probability of the coming event from each of the possible causes, and finally determining the total probability by a process of summation.

CASE VI.—In 600 trials an event occurred 100 times and failed 500 times, what probability thence arises that in 6,000 more trials the number of occurrences of the event shall be contained within the limits 1,000±50 inclusive?

Here put $a=100$, $b=500$; $\alpha=1000$, $\beta=5000$; $x=50$. Then

$$c=\frac{\alpha+\beta}{2\alpha\beta}\div\frac{a}{a+\alpha};\quad t=(x+\tfrac{1}{2})\sqrt{c};\quad \mathrm{P}t=\text{probability}.$$

Calculation.

$\alpha+\beta$	6,000	log 3·77815
$2\alpha\beta$	10,000,000	log 7·00000 sub.
		6·77815
$\frac{a}{a+\alpha}=\frac{1}{11}$		log 8·95861
		log c 5·73676
		log $\sqrt{c}$ 7·86838
$x+\frac{1}{2}=50·5$		log 1·70329
$t=0·3730$		log t 9·57167
P= ·4031, the required probability.		

EXAMPLE 2. In 6,000 trials an event occurred 1,000 times and failed 5,000 times, what is the probability thence to be inferred, that in 6,000 more trials the number of occurrences of the event shall fall within the limits 1,000±50, inclusive?

$\alpha+\beta$	6000	log 3·77815
$2\alpha\beta$	10,000000	log 7·00000 sub.
		6·77815
	$\frac{a+a}{a}=\frac{1}{2}$	log 9·69897
		log c 6·47712
		log $\sqrt{c}$ 8·23856
	$x+\frac{1}{2}=50·5$	log 1·70329
	$\begin{cases} t=0·8747 \\ \mathrm{P}=·7839 \end{cases}$	log t 9·94185

(Compare these examples with that of Case III.)

The methods of computation shown in the foregoing cases must not be considered as limited to mere games of chance. They will apply generally to all kinds of statistical inquiries. Their capability of being directed to useful practical questions connected with business matters is elucidated in the following interesting and concluding example.

Example.—According to the Life Office returns, upon which the "New Experience" life tables are founded, out of 41,385 lives (male and female) age 41, there were 445 deaths, giving 40,940 survivors at the expiration of one year:—

1. Required the chance, deduced therefrom, that for age 41 the true probability of dying in one year is contained between the limits $\frac{445\pm30}{41385}$, inclusive?

2. If another set of 41,385 lives be taken at the same age 41, what is the probability that the number of deaths in one year will fall within the limits 445 ± 30, inclusive?

3. Assuming the observed probability of dying in one year, viz.: $\frac{445}{41385}=\frac{1}{93}$, to be the true rate for that year of age; what then would be the probability that out of another set of 41,385 lives aged 41, the deaths in one year would be contained within the limits 445 ± 30?

1. See Case V.

$\alpha=445$; $\beta=40,940$; $x=30$.

2α	890	log 2·94939
β	40,940	log 4·61215

$2\alpha\beta$	$\log 7{\cdot}56154$ sub.	
$\alpha+\beta$ 41,385	$\log 4{\cdot}61684$	
	$\log c$ $7{\cdot}05530$	
	$\log\sqrt{c}$ $8{\cdot}52765$	
$x=30$	$\log 1{\cdot}47712$	
$t=1{\cdot}0110$	$\log t$ $0{\cdot}00477$	
$P=\cdot 8472$		

2. See CASE VI.

$a=\alpha=445$; $b=\beta=40{,}940$; $x=30$.

2α 890	$\log 2{\cdot}94939$
β 40,940	$\log 4{\cdot}61215$
$2\alpha\beta$	$\log 7{\cdot}56154$ sub.
$\alpha+\beta$ 41,385	$\log 4{\cdot}61684$
	$7{\cdot}05530$
$\dfrac{a}{a+\alpha}=\dfrac{1}{2}$	$\log 9{\cdot}69897$
	$\log c$ $6{\cdot}75427$
	$\log\sqrt{c}$ $8{\cdot}37713$
$x+\frac{1}{2}=30{\cdot}5$	$\log 1{\cdot}48430$
$t=0{\cdot}7268$	$9{\cdot}86143$
$P=\cdot 6960$	

3. See CASE III.

$\alpha=445$; $\beta=40{,}940$; $x=30$.

2α 890	$\log 2{\cdot}94939$
β 40,940	$\log 4{\cdot}61215$
$2\alpha\beta$	$\log 7{\cdot}56154$ sub.
$\alpha+\beta$ 41,385	$\log 4{\cdot}61684$
	$\log c$ $7{\cdot}05530$
	$\log\sqrt{c}$ $8{\cdot}52765$
$x+\frac{1}{2}=30{\cdot}5$	$\log 1{\cdot}48430$
$t=1{\cdot}0279$	$\log t$ $0{\cdot}01195$
$P=\cdot 8540$	

The probability of No. 1 is nearly equal to that of No. 3, and would obviously be identical with the latter if the limits were only slightly extended to $\frac{445+30\frac{1}{2}}{41385}$. Thus we find that the formulæ of Cases III and V suggest the following interesting theorem.

THEOREM. If in $\alpha+\beta$ observations an event has been observed to happen α times and to fail β times, the probability that the true chance of the event lies between the limits $\frac{\alpha\pm(x+\frac{1}{2})}{\alpha+\beta}$ is equal to the probability that in $\alpha+\beta$ observations the number of occurrences of the event shall lie between the limits $\alpha\pm x$, assuming in the latter case that the true chance of the event is known to be $\frac{\alpha}{\alpha+\beta}$.

In the two cases involved in this theorem, the difference of $\frac{1}{2}$ in the values of x is deducible from the consideration that the ordinate (y) of the curve of facility of error may be regarded as practically representing the probability that an error shall fall within the limits $x\pm\frac{1}{2}$; and that the middle or maximum ordinate (Y) may similarly represent the probability of the error being comprised within the limits $\pm\frac{1}{2}$.

The principal questions appertaining to the collection, classification, and discussion of statistics are briefly comprehended in the foregoing cases. The practical resolution of these questions only requires a slight knowledge of logarithms of numbers, and is effected by simply following out the method of working exhibited in the various examples. In the several calculations it will be perceived that instead of using negative indices to logarithms, the complements to 10 are put down and used as positive indices, according to the convenient custom of experienced computers in the higher departments of mathematics; thus, in all additions, subtractions, &c., it is only requisite to reject and borrow 10 whenever necessary.

In almost every branch of statistical inquiry the quantities which enter are subject to deviations from the normal values, arising from a combination of innumerable minute circumstances, of which the variations due to each are separately very small. The total change from the mean value is of course produced by the superposition of these minute variations, some of which tend towards an excess and others towards a defect. When the number of observations is small, they are not unlikely to comprise a number of exceptional cases, in which the greater part of the variations repeatedly tend in the same direction, namely, all towards excess or all towards defect, thus producing divergent results in that particular direction; and these exceptional cases may comprise such a proportion of the total observations as to considerably affect the average result. But when the number of observations is large, the occurrence of such excep-

tional variations, in one direction only, becomes highly improbable, and the exceptions necessarily form such an inappreciable proportion of the total facts, that the disturbing effects become practically neutralized. Thus it is that, although but little dependence can be placed in deductions or conclusions drawn from small numbers, there will certainly be found to exist general laws amongst masses of phenomena, of whatever description, physical or moral, which in the invariable and perpetual recurrence of events of the same kind, as the numbers augment, necessarily approximate towards fixed and determinate ratios. It is also evident that every conclusion will have greater weight, and the approximation to normal results must be greatly assisted and expedited, when the more prominent circumstances which tangibly affect the observations admit of being wholly eliminated, or, still better, of being separately classified and tabulated.

In that case the phenomena appertaining to each set of circumstances or conditions would come under separate and distinct investigation, instead of the more complicated phenomena arising from their diversified combinations. If, however, the various conditions, which principally affect the quantities, do not admit of separation, there will be no alternative but to enlarge the number of observations, if such should be possible, until a reasonable presumption is shown that they present a sufficient accumulation of experience for the purpose of eliciting the required results with a very small probability of error.

As a concluding remark I may add that in this paper I have endeavoured, briefly and practically, to explain some of the most useful and important results and applications of the mathematical theory; and that I shall be glad if it should in any degree assist in directing more attention to the neglected logical science of statistics.

On the Law of the Ages at which Life Insurances are effected. By S. C. Chandler, Jr., *Actuary of the Continental Life Insurance Company, of New York.*

[Reprinted from the "*Spectator*," of New York and Chicago.]

If a classification be made of any considerable number of life policies according to age at issue, a cursory inspection will show that the numbers at the different ages are not fortuitous, but are subject to some unknown, though plainly marked, mathematical relation; and the amenity of the results to law will be more evident

as the number of policies employed in the classification is larger, and the effect of merely accidental fluctuations is consequently more nearly eliminated.

So far as my information extends, no investigation of the character of this law has ever been undertaken. Yet the subject seems to me worthy of consideration aside from the interest that attaches to it simply as a statistical inquiry. The development of such a distributive law would supply actuaries with a very efficient tool in many computations where, a high degree of precision not being essential, the approximate determinations afforded by its application would possess quite as much practical value as the more accurate results of laborious classifications or *seriatim* calculations.

Some attention was given to the problem about two years since, but the material then at hand was too meagre for a definite solution. The publication subsequently, however, of the "Mortality Experience of Life Assurance companies," giving in detail the observations collected by the Institute of Actuaries, supplied abundant data for the desired investigation. To present the results, so far attained, of a discussion of this material, is the object of this paper.

The total number of lives embraced in the statistics of the twenty companies contributing their experience was 160,426. Of this number 11,146 were diseased lives, and 2,433 were exposed to extra risk from climate and occupation. Excluding these two classes, insured under what may be considered exceptional conditions, there remain 146,847 as the number of healthy lives of both sexes insured at all ages. In the *seventh* column of the annexed table is shown the distribution according to current age at entry. It will be seen that, disregarding minor fluctuations, the numbers first increase, and afterwards diminish, at varying rates, the maximum occurring about age 30. The course of the numbers is graphically exhibited in the accompanying chart in which the years of age are laid off on the axis of abscissas, the numbers insured on the axis of ordinates. The tendency towards a continuous curve is obvious.

Now if the number insured at each age is a function of the age, the area of that portion of the curve terminated by any ordinate will likewise be a function of the age. That is, representing by M the number insured at and over the age x,

$$M = f(x).$$

The form of $f(x)$ being found, the number at any given year of age will be the finite difference.

Distribution by Age of Insured Lives.

Current Age at Entry (1)	TWENTY ENGLISH AND SCOTCH OFFICES. No. at and over given age. By formula. (2)	Actual. (3)	Difference. (4)	Ratio. (5)	No. at given age. By formula. (6)	Actual. (7)	Difference. (8)	Ratio. (9)	CONTINENTAL LIFE, OF NEW YORK. No. at given age. By formula. (10)	Actual. (11)	Difference. (12)	Age nearest Birthday. (13)
1	....	146,847	....	....	....	87	....	....	....	....	..	..
2	....	146,760	....	....	....	42	....	....	....	...	..	..
3	....	146,718	....	....	....	39	....	....	....	....	..	..
4	....	146,679	....	....	....	42	....	....	....	....	..	..
5	....	146,637	....	....	....	54	....	....	....	....	..	..
6	....	146,583	....	....	....	69	....	....	....	....	..	..
7	....	146,514	....	....	....	86	....	....	....	....	..	..
8	....	146,428	....	....	....	100	....	....	....	....	..	..
9	....	146,328	....	....	....	140	....	....	....	...	..	..
10	....	146,188	....	....	....	144	....	....	....	....	..	..
11	....	146,044	....	....	....	147	....	....	....	1	..	11
12	....	145,897	....	....	....	156	....	....	....	4	..	12
13	....	145,741	....	....	....	147	....	....	....	5	..	13
14	....	145,594	....	....	....	251	....	....	....	13	..	14
15	....	145,343	....	....	....	296	....	....	....	17	.	15
16	....	145,047	....	....	....	359	....	....	....	17	.	16
17	....	144,688	....	....	....	453	....	....	....	23	..	17
18	....	144,235	....	....	....	606	....	...	...	64	..	18
19	....	143,629	....	....	..	851	....	....	....	102	..	19
20	142,778	142,778	0	·000	728	1,309	−581	−·798	53	133	−80	20
21	142,050	141,469	+581	+·004	1,865	1,779	+ 86	+·046	189	190	− 1	21
22	140,185	139,690	+495	+·004	2,790	2,763	+ 27	+·010	260	313	−53	22
23	137,395	136,927	+468	+·003	3,563	3,371	+192	+·054	332	355	−23	23
24	133,832	133,556	+276	+·002	4,207	3,880	+327	+·078	392	381	+11	24
25	129,625	129,676	− 51	−·004	4,725	4,559	+166	+·035	440	449	− 9	25
26	124,900	125,117	−217	−·002	5,130	4,981	+149	+·029	478	469	+ 9	26
27	119,770	120,136	−366	−·003	5,430	5,322	+108	+·020	506	527	−21	27
28	114,340	114,814	−474	−·004	5,637	5,711	− 74	−·013	525	515	+10	28
29	108,703	109,103	−400	−·004	5,759	5,645	+114	+·020	537	495	+42	29
30	102,944	103,458	−514	−·005	5,806	6,242	−436	−·075	541	572	−31	30
31	97,138	97,216	− 78	−·001	5,792	5,800	− 8	−·001	539	452	+87	31
32	91,346	91,416	− 70	−·001	5,725	5,703	+ 22	+·004	534	526	+ 8	32
33	85,621	85,713	− 92	−·001	5,612	5,598	+ 14	+·002	523	526	− 3	33
34	80,009	80,115	−106	−·001	5,459	5,611	−152	−·028	508	516	− 8	34
35	74,550	74,504	+ 46	+·001	5,281	5,432	−151	−·029	493	526	−33	35
36	69,269	69,072	+197	+·003	5,084	5,222	−138	−·027	473	472	+ 1	36
37	64,185	63,850	+335	+·005	4,866	4,713	+153	+·032	454	454	0	37
38	59,319	59,137	+182	+·003	4,638	4,785	−147	−·032	432	444	−12	38
39	54,681	54,352	+329	+·006	4,410	4,303	+107	+·024	411	396	+15	39
40	50,271	50,049	+222	+·004	4,174	4,530	−356	−·085	389	453	−64	40
41	46,097	45,519	+578	+·013	3,939	3,985	− 46	−·012	367	371	− 4	41
42	42,158	41,534	+624	+·015	3,706	3,567	+139	+·038	345	369	−24	42
43	38,452	37,967	+485	+·013	3,479	3,353	+126	+·036	324	323	+ 1	43
44	34,973	34,614	+359	+·010	3,255	3,156	+ 99	+·031	304	278	+26	44
45	31,718	31,458	+260	+·008	3,038	3,078	− 40	−·013	283	319	−36	45
46	28,680	28,380	+300	+·010	2,829	2,776	+ 53	+·019	263	245	+18	46
47	25,851	25,604	+247	+·010	2,628	2,497	+131	+·050	245	240	+ 5	47
48	23,223	23,107	+116	+·005	2,436	2,344	+ 92	+·037	227	207	+20	48
49	20,787	20,763	+ 24	+·001	2,251	2,245	+ 6	+·003	210	232	−22	49
50	18,536	18,518	+ 18	+·001	2,074	2,158	− 84	−·041	193	232	−39	50

Distribution by Age of Insured Lives (continued).

Current Age at Entry.	Twenty English and Scotch Offices.								Continental Life of New York.			Age nearest Birthday.
	No. at and over given age.				No. at given age.				No. at given age.			
	By formula.	Actual	Difference.	Ratio.	By formula.	Actual.	Difference.	Ratio.	By formula.	Actual.	Difference.	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
51	16,462	16,360	+102	+.006	1,903	1,868	+ 35	+·019	177	211	−34	51
52	14,559	14,492	+ 67	+·005	1,742	1,745	− 3	−·002	163	151	+12	52
53	12,817	12,747	+ 70	+·005	1,588	1,558	+ 30	+·019	148	156	− 8	53
54	11,229	11,189	+ 40	+·004	1,442	1,344	+ 98	+·068	134	131	+ 3	54
55	9,787	9,845	− 58	−·001	1,304	1,248	+ 56	+·044	122	120	+ 2	55
56	8,483	8,597	−114	−·014	1,173	1,141	+ 32	+·028	109	94	+15	56
57	7,310	7,456	−146	−·020	1,050	1,040	+ 10	+·010	98	65	+33	57
58	6,260	6,416	−156	−·025	935	902	+ 33	+·035	87	76	+11	58
59	5,325	5,514	−189	−·035	827	889	− 62	−·075	77	70	+ 7	59
60	4,498	4,625	−127	−·028	726	871	−145	−·200	68	66	+ 2	60
61	3,772	3,754	+ 18	+·005	633	600	+ 33	+·053	59	37	+22	61
62	3,139	3,154	− 15	−·005	548	533	+ 15	+·027	51	38	+13	62
63	2,591	2,621	− 30	−·011	471	464	+ 7	+·015	43	28	+15	63
64	2,120	2,157	− 37	−·018	401	392	+ 9	+·022	38	31	+ 7	64
65	1,719	1,765	− 46	−·027	338	369	− 31	−·092	31	12	+19	65
66	1,381	1,396	− 15	−·011	283	288	− 5	−·018	27	12	+15	66
67	1,098	1,108	− 10	−·009	233	239	− 6	−·026	21	14	+ 7	67
68	865	869	− 4	−·005	191	207	− 16	−·084	18	5	+13	68
69	674	662	+ 12	+·018	155	143	+ 12	+·078	15	5	+10	69
70	519	519	0	·000	124	119	+ 5	+·041	11	2	+ 9	70
71	395	400	− 5	−·012	97	79	+ 18	+·185	37	0	+37	71 and over.
72	298	321	− 23	−·077	77	84	− 7	−·091				
73	221	237	− 16	−·072	59	53	+ 6	+·102				
74	162	184	− 22	−·136	44	55	− 11	−·250				
75	118	129	− 11	−·093	34	33	+ 1	+·029				
76	84	96	− 12	−·143	25	17	+ 8	+·320				
77	59	79	− 20	−·339	18	15	+ 3	+·167				
78	41	64	− 23	−·561	13	14	− 1	−·077				
79	28	50	− 22	−·786	9	10	− 1	−·111				
80 and over.	19	40	− 21									

There being no direct means of arriving at a knowledge of the form of the required function, we must have recourse to tentative processes. The transcendental expression which has been adopted as best representing the observations is given below. The form is unsatisfactory, and I am strongly induced to believe that it is susceptible of modification and improvement.

It will be observed that of the total number of insurers 4,069, or two and three-quarters per cent, entered below the age of twenty years. Probably a large proportion of these held children's endowments. What influence this fact had in the result there is no means of ascertaining; but all attempts to find an expression which would

accord with the observed entrances below age 20, proved futile. For those above 20 the formula which best represents the observed numbers is

$$M_x = (ab^{-x^k} c^{x^k \sin x\phi}) \quad . \quad . \quad . \quad (1)$$

in which a is the total number insured at all ages; x the excess of the age above 20 years; and b, c, k, and ϕ, are constants dependent upon the observations themselves. This equation, although entirely an empirical one, gives results which present a close agreement with the recorded numbers at each age.

Taking the logarithms of both members of (1) and putting

$$\mu = \log M, \qquad \beta = \log b,$$
$$\alpha = \log a, \qquad \gamma = \log c,$$

we may write

$$\mu = \alpha - (\beta - \gamma \sin x\phi) x^k. \quad . \quad . \quad (2)$$

Differentiating with respect to β, γ, ϕ, and k, we obtain

$$\left.\begin{array}{l} x^k d\beta - x^k \sin x\phi \, d\gamma + x^k \dfrac{\lambda x}{\lambda e} (\beta - \gamma \sin x\phi) dk \\ \qquad - x^{1+k} \cos x\phi \, d\phi + d\mu = 0 \end{array}\right\} \quad . \quad . \quad (3)$$

where λe is the modulus of the common system of logarithms, or the logarithm of the Naperian base.

Rough determinations of the constants b, c, ϕ, and k, first having been obtained by a combination of graphical and numerical processes, the values of the differential coefficients and the absolute term of (3) were calculated for twelve different values of x. The solution of the resulting equations gave three independent determinations of each of the corrections $d\beta$, $d\gamma$, dk, and $d\phi$, the mean of which being applied to the assumed values, furnished a second approximation to the constants. These again, being substituted in (2) and (3) for all values of x, final corrections were deduced from the equations of condition by the method of least squares.

In the solution a certain indeterminateness, arising from the nature of the equations of condition, becomes apparent; the values of the corrections depending greatly upon the number of equations employed in the formation of the normal equations. This fact and the systematic deviations shown by the residuals, confirm me in the conviction that the equation (1) does not represent the true law of which we are in search, though affording a close approximation to it. The adopted values of the constants given below, were obtained by first determining $d\beta$ and $d\gamma$ in terms of dk and $d\phi$ from the normal equations in $d\beta$ and $d\gamma$, using 55 equations of condition

(embracing ages 21 to 75 inclusive), and then introducing the values dk and $d\phi$ derived, by an exercise of judgment, from a consideration of the results of several groupings of the equations of condition.

Applying the corrections in this way deduced, we find

$$b=1{\cdot}0052,$$
$$c=1{\cdot}00159,$$
$$k=1{\cdot}86,$$
$$\phi=2°30';$$

and these adopted values, and that of $a=142{,}778$ substituted in (1) give

$$M=142778\times 1{\cdot}0052^{-x^{1\cdot86}}\times 1{\cdot}00159^{x^{1\cdot86}\sin x(2°.5)},$$

which may also be written

$$M=142778\left(\frac{1{\cdot}00159^{\sin x(2°\cdot5)}}{1{\cdot}0052}\right)^{x^{1\cdot86}} \quad . \quad . \quad . \quad . \quad (4)$$

For a comparison of the formula with actual experience the values of M have been computed by means of (4) and inserted in the *second* column of the table. The first differences, being the numbers insured at the several ages according to the formula, are given in the *sixth* column. The discordances between the computed and the actual numbers, with the proportion which these bear to the former, are exhibited in columns adjacent.

Though the signs of the differences between the computed and the actual values of M show that systematic errors have not been entirely eliminated, the deviations are confined within narrow limits. The average discordance up to age 70, without regard to sign, is less than one per cent (0·008); while in but one instance (age 59) is the discordance so large as three per cent of the computed value of M.

Turning to the number insured at each age, it will be seen that the formula presents a very satisfactory accordance with experience. The signs in the column of differences seem immethodical, there being thirty-two changes of sign between 20 and 87, the age at which the number insured becomes zero by the formula; while the more prominent discrepancies are traceable, on reasonable grounds, to a sufficient cause.

To facilitate comparison of the calculated with the observed numbers, the values in the *sixth* column have also been plotted in the chart already referred to, and connected by the dotted line. The

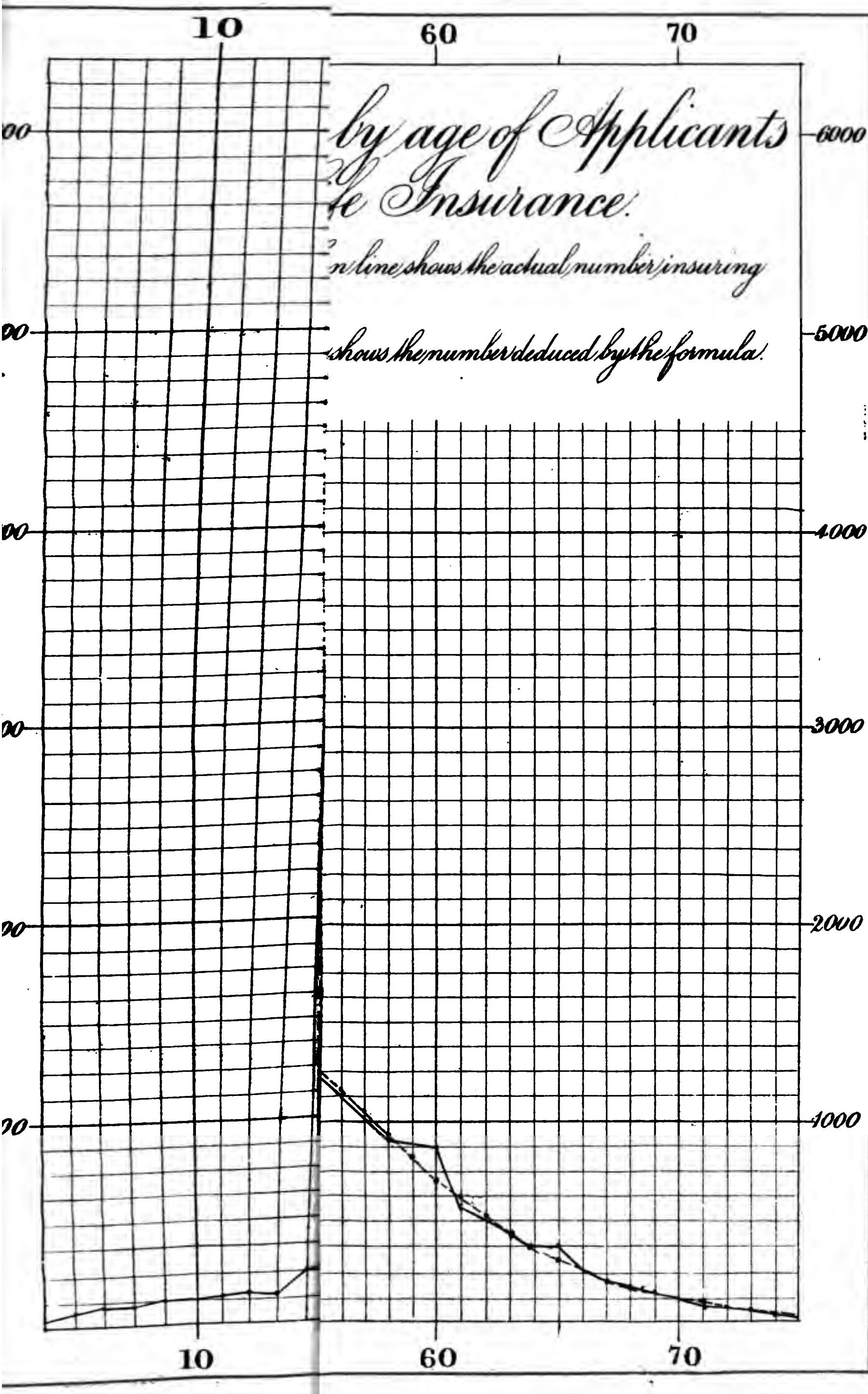
near coincidence of the resulting curve with the continuous broken line representing the recorded numbers is striking. For ages over 75 the two are appreciably identical, on the scale of the chart, and have therefore been omitted.

Considerable instruction may be derived from an examination of the residual errors. In following down column *eight* the eye is attracted by large negative residuals at ages 20, 30, 40, 50, and 60. The corresponding effect in the chart is a prominence at each of the ages mentioned. A similar effect, though less noticeable, is exhibited at 35, 45 and 65. There would thus seem to be a tendency to excess of the actual numbers over those indicated by theory at ages which are multiples of five years; a tendency more marked at those ages which are likewise multiples of ten years.

Now, does this peculiarity arise from a real excess of applicants for insurance at the ages indicated, or from a tendency to mis-statement such as would lead persons slightly older or younger than, say, for instance, 30 years, to represent themselves from ignorance, carelessness or intention, as exactly 30 years of age? Mr. Nathan Willey has suggested that there may be certain epochs in life, in approaching which men are reminded of the ebb of their existence, and take more readily and seriously into consideration the subject of provision for their families against their death. This hypothesis is not, perhaps, altogether a fanciful one, and the phenomena observed may possibly be due in some part to such a cause. The more probable explanation, however, it seems to me, is the second above proposed. A similar interpretation of analogous irregularities in the statistics of our volunteer soldiery in the late war, has been given by that distinguished authority, Dr. B. A. Gould.*

It becomes interesting to ascertain how nearly the formula deduced from the experience of the English offices—of which one began operations one hundred and fifty years ago, and the youngest was nineteen years old at the time the data were collected—represents the relative proportions of applicants for insurance at different ages in American companies. Some material for a comparison is readily available in the classifications, made for valuation purposes, of policies issued by the company with which I am connected. Since it would be manifestly improper, in such comparison, to include endowment insurances, I have taken simply the life policies issued from January 1, 1869, to July 1, 1871. The distribution

* Ages of the United States volunteer soldiery. Statistical Bureau, United States Sanitary Commission. New York, 1866.



10
60
70
by age of Applicants
e Insurance.
n line shows the actual number insuring
shows the number deduced by the formula.
6000
5000
4000
3000
2000
1000
10
60
70

of these according to age at issue is exhibited in the *eleventh* column of the table; and in the *tenth* column the corresponding numbers* found from the formula (4), substituting the value of $a = 13,304$ in place of 142,778. The adjacent column of differences shows that there is no systematic deviation below age 55. At ages over 55 the number issued is considerably smaller than that indicated by the formula. This result is referable in great measure, I believe, to the caution exercised in accepting risks upon old lives.

It will not fail to be observed that the same peculiarities of the discordances at particular ages noticed in the English experience, are here more prominently marked.

As a whole the result of the comparison is to strengthen the probability of the prevalence of a general law relative to the ages of applicants for life insurance.

Extract from Lord Neaves's Opening Address as President of Section F, (Economic Science and Statistics) at the Meeting of the British Association at Edinburgh, 1871.

DEDUCTIONS FROM THE REGISTRAR'S RETURNS.

I FEEL called upon to say that I consider our Registers in Scotland to be, generally speaking, in a most satisfactory state, particularly in the important department of Vital statistics, as to which the reports of the Registrar-General, embodying the reports made to him by Dr. Stark, contain reliable information of the most interesting and important kind. One singular result that seems to have been established by the tables there given is, that at every quinquennial period of life from 20 years of age up to 85, married men die in Scotland at a much lower rate than unmarried men. Sometimes the difference is very great, particularly between 20 and 45, up to which period it approximates to as high a rate as 2 to 1; but after that, the difference, though less, is still very considerably in favour of the married men. The subject is more complicated as regards women, from obvious causes; though here, too, marriage seems to be the more favoured state. As regards both sexes, the advantage on the side of marriage is easily accounted for up to a certain point. Generally speaking, those

* The difference of practice in classification, according to age, at *nearest* or *next* anniversary of birthday, does not materially affect the results of the comparison.

who marry are likely as a class to be better lives than those who do not. The unmarried will infallibly include a greater number of sickly or diseased constitutions than the married class. Without professing myself an implicit believer in Darwin, I acknowledge the truth of several of his statements in his "Descent of Man," as to what he calls Sexual selection. As a general rule, the attachments that lead to marriage will be prompted by considerations that are intimately connected with health and strength. Good looks, cheerful tempers, and buoyant constitutions are great attractions, and those who are wholly devoid of these, as well as those who are the victims of positive bad health, will often be excluded from having tickets in the matrimonial lottery. No doubt, causes occur not unfrequently which disturb these natural tendencies. Some of these causes are allowable or laudable, others are the reverse. In a few cases affection leading to marriage may be inspired by great virtue, or great talent, or high accomplishments, though not associated with health or strength. In other cases, connections may be formed that are wholly unconnected with love—as where rank, or wealth, or influence may overcome the natural repugnance excited by deformity or disease. Burns, I think it is, that says—

> "Be a lassie ne'er sae black,
> If she hae the penny siller,
> Set her upon Tintock tap—
> The wind will blaw a man till her."

Still, as a general rule, both men and women who are married are likely, on an average, to have more health and vitality than those who remain single. As regards the male sex, again, those of them that are of dissolute habits or unsettled and thriftless dispositions, are not so likely to marry as those who are orderly and well-conducted, and in favourable circumstances of life. But after making allowance for these elements, it still appears that the death-rate of married men is at all periods of life lower than that of the unmarried. This can be accounted for only on the footing that marriage is favourable to health, by conducing to regular habits of life, and by giving natural scope to the domestic affections. It cannot be doubted, for instance, that an old man who has a wife to take care of him, will be much better looked after than if he lived alone. It is not necessary in adopting this view, to suppose that the married life is to be wholly free from sorrows, cares, and anxieties. Even these are not always prejudicial to health; and we are, perhaps, the better for them when they are well encountered. Neither is it essential that the matrimonial current should always

run a smooth course. Most of us, probably, would agree with the view taken by Paley, who, when an old clergyman at an episcopal dinner asserted that he had been married for forty years, but had never had a difference with his wife, observed quietly to the bishop that "it must have been very flat." An occasional ripple will occur in all water, unless it be frozen over, and perhaps after marriage, as well as before it, there may be truth in the maxim, "Amantium iræ amoris redintegratio."

In referring to this matter, it has occurred to me to consider whether, if the lower death-rate of married persons is an ascertained fact, this may not partly account for the general success of Life Insurance Offices when well conducted. It is clear that an Office transacting on the usual calculations of mortality, has advantages of various kinds. In particular, its medical examinations, which are a most important part of its constitution, exclude hazardous lives, except, at least, at extra premiums. The rank of life, probably, of parties effecting insurances may also benefit the Office; but if married men are to a certain extent to be considered as selected lives, this also, I should think, must tell in favour of the Office, as I presume that, from family reasons, more married men effect insurances than unmarried men.

HOME AND FOREIGN INTELLIGENCE.

[The publication of the Returns under the Life Assurance Companies Act, 1870, will involve some alteration in the manner of presenting the Bonus Reports which we have hitherto pursued. We therefore purpose, in future, to extract from those Returns certain items of information which are not contained in the Bonus Reports. In particular, we think it will be useful to give the Consolidated Revenue Account, the Summary and Valuation, and the Valuation Balance Sheet, of the Fifth Schedule of the Act, these Returns often showing more clearly than the Bonus Report the manner in which the surplus divided has been arrived at.—Ed. *J. I. A.*]

GRESHAM LIFE ASSURANCE SOCIETY.

Established 1848.

Report of the Actuary

On the Society's contracts for Assurances and Annuities in force on the 30th June, 1870.

I have examined the registers of the Society, and extracted from them all the data requisite for a valuation. As for the policies on the lives of

adults, the valuation has been made on the basis of the Law of Mortality deduced from the experience of seventeen Life Assurance Offices; while for those depending upon the lives of children, the Carlisle tables of mortality have been employed because the experience tables commence only at the age of ten years. The rate of interest assumed in the calculations was at the rate of 3½ per cent.

* * * * *

Taking into consideration the general arrangements of the Society, I am of opinion, and recommend that the Board should reserve a sum of £600,000 out of the present value of the premium income to provide for future expenses and future bonuses.

The Assurance and Annuity Fund would in that case be as follows:—

Value of sums assured and bonus, less sums re-assured		£4,866,868
Value of premium income, less the amounts paid for re-assurances	£4,260,853	
Reserve for future expenses and bonuses	600,000	
		3,660,853
Assurance Fund		£1,206,015
Value of sums payable for annuities	£127,134	
Value of the premium income receivable thereunder	2,428	
Annuity Fund		124,706
		£1,330,721

This total of £1,330,721 is required as a reserve for the assurance and annuity contracts in force on the 30th June, 1870, and should appear as such in the General Balance Sheet of the Society for that date.

GENERAL BALANCE SHEET, FOR THE 30TH JUNE, 1870.

Dr.	£	*s.*	*d.*
Assurance and Annuity Fund	1,330,721	0	0
Shareholders' Capital fully subscribed £100,000, and paid thereon	21,712	0	0
Claims admitted, waiting settlement	28,583	16	7
Other Accounts waiting settlement in due course	5,102	5	6
Balance	66,893	7	3
	£1,453,012	9	4

INVESTMENTS:— *Cr.*	£	*s.*	*d.*	£	*s.*	*d.*
Government Funds—						
English Consols and New Three per Cents, &c.	293,490	0	0	271,732	2	0
French Rentes, Three per Cent.	32,666	13	4	22,771	6	6
Algerine Bonds, Four per Cent.	11,028	0	0	9,998	14	5
Italian Rentes, Five per Cent.	91,828	0	0	57,594	19	11
Belgian Rentes, Four-and-a-Half per Cent.	8,375	19	0	8,629	6	5
Bavarian State Bonds	14,806	6	9	14,284	15	4
Baden State Bonds	30,428	6	10	27,517	2	3
Dutch and other Bonds	3,958	4	0	3,523	16	2
Austrian Exchequer Bills and Public Mortgage Funds	33,482	9	10	33,482	9	10

Railway Securities, with Government Guarantee—						
The Great Indian Peninsular Railway Debenture Stock, 4 per Cent.	17,350	0	0	17,382	18	6
East Indian Railway Convertible Debentures, 5 per Cent.	11,300	0	0	12,682	11	11
Great Southern of India Railway Consolidated Stock, 5 per Cent.	5,000	0	0	5,342	4	0
East Bengal Railway Stock, 5 per Cent.	7,000	0	0	7,671	9	6
Madras Railway Stock, 5 per Cent.	7,000	0	0	7,627	10	0
Great Western of Canada Railway Bonds, 5½ per Cent.	9,600	0	0	9,028	0	0
The Western Railway of Austria Preference Bonds	24,816	0	0	23,989	6	9
„ Carl Ludwig ditto ditto	8,016	0	0	7,394	15	2
„ North Western ditto ditto	8,000	0	0	7,160	0	0
„ Sud Bahn ditto ditto	8,048	0	0	7,401	10	0
„ Linz Budweiser ditto ditto	8,000	0	0	7,432	0	0
„ Kaschau Oberberger ditto ditto	8,000	0	0	7,242	8	0
„ Eastern Railway of Hungary ditto	24,000	0	0	20,400	0	0
Bavarian Eastern Railway Bonds	4,950	0	0	5,152	14	7
Swiss North Eastern Railway Shares	2,126	5	0	2,434	3	0
Obligations of the Paris, Lyons, and Med. Railway Fusion, 1866	67,420	0	0	46,831	14	4
Freehold Property—						
Boulevard des Capucines, No. 8, Paris				51,141	3	1
Rue Halevy, No. 2 do.				31,587	3	9
Rue de Provence, No. 30 do.				45,000	0	0
Friedrichstrasse, No. 36, Mannheim				5,588	4	0
Leaseholds—in London				9,437	3	4
Mortgages—						
Of Real Estate				172,687	5	5
Other Mortgages, including Life Interests and Reversions				63,248	16	7
Corporation of the City of London Bonds				31,062	10	0
Advances to Corporations of various Towns of Great Britain				108,178	17	3
Policies within the Surrender value thereof				113,281	9	3
On Personal Security				30,635	9	0
Agents Balances including Premiums due and not received				101,350	2	4
Town Department do. do.				1,045	8	3
Interest accrued and not received				15,331	9	0
Cash—at Bankers	9,999	8	6			
on deposit	13,297	3	2	23,368	2	3
in hand	71	10	7			
Furniture and Fittings at Head and Branch Offices				3,598	0	0
Rent due				1,763	7	3
				£1,453,012	9	4

It will be observed that the total assets of the Society realised up to the date of the Balance Sheet amounted to £1,453,012 9*s.* 4*d.*, and that setting against this amount the funds reserved for the Assurance and Annuity Contracts, the claims waiting settlement, some other accounts, and the share capital paid up, there remains an excess of asset over liability of £66,893 7*s.* 3*d.*

Under ordinary circumstances the whole of this balance might be considered available for distribution among the members entitled to participate, but looking to the fact of the present disturbed state of the Continent of Europe, the Directors restrict the amount to £50,000 which they now declare as the sum to be apportioned among the members by way of bonus: whilst the remaining sum of £16,823 9*s.* 3*d.* raises the general fund for Assurances and Annuities, to the sum of £1,347,614 7*s.* 3*d.*

Eighty per cent. of the amount declared as bonus will be apportioned to the holders of participating policies in force at the date of the Balance Sheet, and will become applicable on the 30th of June next, to such policies as shall have been in force three full years: or later, when the three years from the date of the policy shall have been completed. In the case of claims under participating policies, of not less than three years duration, by death happening prior to the 30th June next, the cash value of the bonus to be allotted will be allowed. In other cases the usual options will be allowed to the policy holders, in selecting the mode in which they desire that their bonus may be applied, but subject, in all respects, to the regulations of the Society, which will be stated in the notices to be issued to the Members, when the requisite calculations in detail shall have been completed.

* * * * *

PARTICULARS OF THE ASSURANCE AND ANNUITY CONTRACTS

In force on the 30th June 1870.

WHOLE LIFE POLICIES—WITH PROFITS.

Present Ages.	Number of Policies.	Sum Assured, with Bonus Additions.	Office Premiums.	Value of Sum Assured and Bonuses.	Value of Office Premiums.
		£	£	£	£
14 to 20	14	4,100	79	1,167	1,588
21 „ 25	171	81,504	1,742	25,326	33,757
26 „ 30	761	408,075	9,316	137,953	173,098
31 „ 35	1,507	798,335	20,280	294,764	358,008
36 „ 40	1,968	912,621	25,880	371,837	427,842
41 „ 45	2,234	1,082,158	34,634	490,679	525,476
46 „ 50	2,098	974,133	34,368	482,158	472,913
51 „ 55	1,546	715,887	28,999	398,549	353,861
56 „ 60	929	400,714	19,009	245,781	200,283
61 „ 65	522	218,359	12,177	144,021	108,114
66 „ 70	230	92,684	5,399	67,913	38,952
71 „ 75	92	33,359	2,274	26,379	12,723
76 „ 80	19	6,587	513	5,527	2,160
81 „ 85	7	1,247	144	1,216	330
	12,098	£5,729,727	£194,814	£2,693,270	£2,709,105

ENDOWMENT ASSURANCE POLICIES—WITH PROFITS.

Age to receive at.	Number of Policies.	Sum Assured, with Bonus Additions.	Office Premiums.	Value of Sum Assured and Bonuses.	Value of Office Premiums.
		£	£	£	£
45	135	53,883	3,136	36,196	29,284
50	1,229	510,276	25,738	319,066	270,601
55	1,204	532,704	27,369	335,360	283,442
60	957	461,236	23,670	289,845	241,821
65 and upwards.	605	223,839	16,548	186,637	163,359
	4,130	£1,851,938	£96,461	£1,167,104	£988,507

ENDOWMENT POLICIES—WITH PROFITS.

Age to receive at	Number of Policies.	Sum Assured, with Bonus Additions.	Office Premiums.	Value of Sum Assured and Bonuses.	Value of Office Premiums.
		£	£	£	£
14 to 21	4,195	834,595	32,572	556,024	224,385
25 „ 70	1,202	305,221	11,336	189,342	75,707
	5,397	£1,139,816	£43,908	£745,366	£300,092

MISCELLANEOUS—WITH PROFITS.

	Number of Policies.	Sum Assured with Bonus Additions.	Office Premiums.	Value of Sum Assured and Bonuses.	Value of Office Premiums.
		£	£	£	£
Joint Assurances	281	94,998	4,612	50,700	57,812
Contingent and limited payments	74	51,887	1,940	20,027	20,449
	355	£146,885	£6,552	£70,727	£78,261

WHOLE LIFE POLICIES—WITHOUT PROFITS.

Present Ages.	Number of Policies.	Sum Assured.	Office Premiums.	Value of Sum Assured.	Value of Office Premiums.
		£	£	£	£
15 to 20	1	160	3	46	56
21 „ 25	17	6,936	133	2,141	2,652
26 „ 30	48	33,440	691	11,208	13,243
31 „ 35	93	52,736	1,231	19,393	22,383
36 „ 40	150	68,457	1,754	27,838	29,865
41 „ 45	147	57,297	1,658	25,878	26,019
46 „ 50	149	55,036	1,887	27,657	26,719
51 „ 55	160	63,190	2,415	35,084	30,588
56 „ 60	124	57,231	2,526	34,884	27,865
61 „ 65	123	49,281	3,094	32,741	29,166
66 „ 70	62	28,491	1,950	20,518	15,005
71 „ 75	8	3,220	254	2,468	1,630
76 „ 80	7	2,430	178	1,981	882
81 „ 87	2	300	30	260	101
	1,091	£478,205	£17,804	£242,097	£226,174

MISCELLANEOUS—WITHOUT PROFITS.

	Number of Policies.	Sum Assured.	Office Premiums.	Value of Sum Assured.	Value of Office Premiums.
		£	£	£	£
Endowment Assurances ..	17	3,800	197	2,426	2,305
Joint " ..	11	2,809	147	1,770	1,412
Term " ..	824	99,475	3,455	3,455	1,727
SURVIVORSHIP— Contingent Limited Payments and Special ..	22	10,520	3,420	3,860	1,446
Total	874	£116,604	£7,219	£11,511	£6,890

SUMMARY OF THE ASSURANCES.

WITH PROFITS.	Number of Policies.	Sum Assured, and Bonus.	Premium.	Value of Sum Assured and Bonuses.	Value of Office Premium.
		£	£	£	£
Whole Life	12,098	5,729,727	194,814	2,693,270	2,709,105
Endowment Assurance ..	4,130	1,851,938	96,461	1,167,104	988,507
Endowments	5,397	1,139,816	43,908	745,366	300,092
Miscellaneous	355	146,885	6,552	70,727	78,261
Total	21,980	£8,868,366	£341,735	£4,676,467	£4,075,965
WITHOUT PROFITS.		£	£	£	£
Whole Life............	1,091	478,205	17,804	242,097	226,174
Miscellaneous	874	116,604	7,219	11,511	6,890
Total	1,965	£594,809	£25,023	£253,608	£233,064
		£	£	£	£
Collective Total	23,945	9,463,175	366,758	4,930,075	4,309,029
Re-Assured	(141)	120,986	4,236	63,207	48,176
Result	23,945	£9,342,189	£362,522	£4,866,868	£4,260,853

ANNUITIES.

Class.	Number.	Amount.	Annual Premium.	Value of Annuity.	Value of Premium.
		£	£	£	£
Immediate Annuities	465	17,964	..	115,401	..
Joint Life	18	603	..	5,780	..
Deferred "	14	576	345	4,589	1,881
Contingent "	7	310	70	1,364	547
Totals..........	504	£19,453	£415	£127,134	£2,428

[From the Chairman's address it appears that the investments are made at an average rate of interest of $4\frac{3}{4}$ percent, yielding about $4\frac{1}{4}$ percent upon the total assets of every kind.]

CROWN LIFE ASSURANCE COMPANY.

Established 1825.

Quinquennial Report,

Shewing the progress of the Company in the past five years, and the result of the Quinquennial investigation of its affairs as they existed on the 25th March, 1870.

I. *Progress of the Business.*

During the five years the Company has issued 4,077 Policies, assuring the sum of £2,175,359, and yielding a New Annual Premium Income of £66,155. In the preceding Quinquennial period there were issued 2,359 Policies for the sum of £1,225,643, and yielding a Premium Income of £37,598.

II. *Position of the Company.*

The total Policies in force on the 25th March, 1870, were 7,697, assuring £4,194,457, Sum Assured, and £230,851 Bonus Additions.

The total Annual Income was, from Premiums, £126,378, and from Interest, £51,996—together, £178,374.

The total Net Funds, comprising the Assurance Fund and the Proprietors' Guarantee Fund, amounted to £1,126,281. (See Balance Sheet.)

III. *Investments.*

The average rate of Interest receivable on the Investments of the Assurance Fund is £4 : 12 : 3 per cent. per annum. The Accounts in the Ledger have been revised, and no Investment stands in the books at a higher price than the market value, while many of them would realise more than the price taken credit for as an Asset in the Balance-Sheet.

IV. *Mortality.*

Claims have accrued in the five years under 635 Policies for the sum of £401,871 sum assured. There has also been paid the sum of £66,173 as Bonus Additions to such of the Policies as were entitled to participate in the Profits.

In the year 1853, considerable changes were made in the Assurance forms, and since that time all proposers have been required to furnish information respecting their family history, which it was thought would enable the Directors to make a better selection of lives; and as the mortality in the last period has been much heavier than was anticipated, it was considered desirable to ascertain whether the increased mortality had occurred among the older Assurances effected before 1853, or among the newer Assurances effected since the adoption of the new forms in that year.

The mortality expected to have arisen in each of the last seventeen years has accordingly been calculated for the Policies issued *before*, and those issued *since* 1853, separately; and the results show that in the newer business the actual claims have been *less* than the expected claims by the sums of £17,345 in the seven years to 1860; by £19,937 in the five years to 1865; and by £32,178 in the last five years to 1870.

In the older business,—the Policies issued *before* the year 1853—the actual claims have *exceeded* the expected amount in each period. In the seven years to 1860, the *excess* was £16,130. In the next five years to 1865, the amount of the older Assurances had, of course, considerably decreased, but, nevertheless, the claims were still in *excess* of the expected amount by £7,391. In the last five years to 1870, notwithstanding that the amount at risk under the older Policies had been much further reduced, the actual claims have *exceeded* the expected amount by the sum of £40,616.

This excessive payment on account of claims has, of course, very materially affected the profit of the period, but the Directors regard the unfavourable result as one of the fluctuations that are frequently met with in the practice of Life Assurance.

With respect to the future, the Directors wish it to be particularly borne in mind that the tabular results, with which the Company's mortality has been compared, were deduced from the Equitable Society's experience of the deaths among its own members; and as there are good grounds for assuming that the older business of the Crown was as carefully selected as that of the Equitable, there is no reason to expect that the mortality among the older members of the Crown will be greater, in a series of periods, than among the members of equal ages that were assured in the Equitable; and it is, therefore, a fair inference, that the excessively adverse result in the last five years will be compensated for in future periods by fluctuations in an opposite and favourable direction.

It will, also, be satisfactory for all interested in the Company to know that the Assurances effected *before* 1853 are now but a small proportion of the whole Assurances at risk, the amount on the 25th March last having been only £879,000, while the Policies effected *since* 1853 amounted, at the same date, to £3,315,000.

V. *Principles of the Valuation.*

The Table of Mortality employed in all the Valuations on this occasion is the same as had been used at all former investigations, namely, the Equitable Experience Table.

The *net* rate of interest realised from the investment of the Assurance Fund during the period has averaged about 4½ per cent; but in valuing the Policies of Assurance the rate assumed to be hereafter realised is only 3½ per cent. The rate assumed in valuing the Annuity transactions, and the Reversionary Bonus on Policies, is 4 per cent. Whatever interest may, in future, be received on the investments beyond these rates, will be profit, to be apportioned at future divisions.

In estimating the liability under the Policies, the whole of the "Loading," amounting to more than £26,000 per annum, has been excluded, and credit has been taken in the account for only the present value of the future *pure* or *risk* Premiums, so that no profit to be hereafter realised has been anticipated. The present value of the "Loading" thus reserved for future profit and expenses exceeds £358,000.

It is upon these data that each Policy has been separately valued, and the results, which have been carefully checked, are given in the Balance Sheet on page 75.

VI. *Distribution of the Surplus Fund.*

				£	s.	d.
The Total Cash Surplus for division is shewn by the Balance Sheet [page 75] to be				82,576	2	8
Of which, in accordance with the provisions in the Deed of Settlement, there have to be applied, in augmentation of the Guarantee Fund, which constitutes a permanent security to the Assured:—						
One-third of the Surplus applicable to to Policies issued *before* 25th March, 1860	£15,492	0	0			
One-sixth of the Surplus applicable to Policies issued *after* 25th March, 1860	5,657	0	0			
One-sixth of the Surplus applicable to the Scottish Friendly Company's Policies	125	0	0			
Share of the Intermediate Bonus paid on Policies that have become claims since 25th March, 1865	1,400	13	2			
				22,674	13	2
Leaving Cash Surplus to be alloted among the Assured whose Policies were in existence on the 25th March, 1870				£59,901	9	6
Which sum will be apportioned in the following manner, namely:—						
Among the Assured under 2,427 Policies for £1,416,972 dated *before* 25th March, 1860				30,985	0	0
Among the Assured under 4,462 Policies for £2,223,692 dated *after* 25th March, 1860				28,289	0	0
Among the Assured under 124 Policies for £43,550 of the Scottish Friendly Company				627	0	0
				£59,901	0	0

The sum of £22,674 : 13 : 2 now to be added to the Guarantee Fund enables the Directors to declare a Bonus of £3 : 13. on each Share, retaining a balance of £190 : 13 : 2. unappropriated. The Fund which in 1865 consisted of 6,160 Shares at £30 : 12, and amounted to £188,496, will, after such apportionment, consist of 6,160 Shares at £34 : 5, and amount to £210,980.

[A dividend is declared for the ensuing year at the rate of 31 shillings per share free of Income Tax, payable by half-yearly instalments on the 15th July and 15th January.]

The sum of £59,901 now to be apportioned among the Assured, represents Reversionary Bonus Additions amounting to £125,670, and the sums assured by the Participating Policies in force on the 25th March last, have accordingly been increased to that extent.

* * * *

VII. *Regulations as to Application of the Bonus.*

The Bonus allotted to Policies on which three or more years' Premiums were paid on the 25th March last immediately *vests* in the Assured, and in the case of Policies of shorter duration the Bonus will become *vested* when the third year's Premium shall have been paid.

If any Policy on which not less than three years' Premiums were paid has become a claim since the 25th March last, the Bonus now declared

The Policyholders have the option of applying the Bonus to augment the sum assured; or they may receive the value of the *vested* Bonus in cash, or apply it towards the reduction of their Annual Premiums, for life, or for five years, until the next division of profits, which will be in the year 1875. A Table, showing the values of the Bonuses, at the various ages, if dealt with in any of these ways, will be printed on the Bonus Certificate, which will be sent to each Policyholder. The Assured may also elect to have the Bonus applied so as to make the sum in the Policy payable to himself if he live to an age to be specified in a Certificate, to be issued at each Quinquennial declaration of profit.

* * * *

BALANCE SHEET of the Company on the 25th March, 1870.

LIABILITIES.

	£	s.	d.
Shareholders' Capital paid up	188,496	0	0
Shareholders' Reserve Fund	902	5	0
Assurance Fund	936,883	13	5
Total Funds	1,126,281	18	5
Claims admitted, but not due	32,478	10	0
Dividends due, but not paid	1,144	16	3
	£1,159,905	4	8

ASSETS.

	On Account of Shareholders' Fund. £	s.	d.	Assurance Fund. £	s.	d.
Mortgages on Real Estate	100,249	12	0	342,148	12	4
„ on the Company's Policies	..			49,891	6	5
INVESTMENTS—						
British Government Securities (£50,000 New Three per Cents.)	..			46,000	0	0
English Railway Debentures	39,975	0	0	250,904	2	0
Indian Railway Debentures	..			12,412	10	0
Company's London, Edinburgh, and Dublin Offices	..			25,327	10	11
Life Interests and Reversions	..			21,705	0	8
London Water Companies' Debentures	13,400	0	0	34,657	18	4
County and Borough Rates	26,305	8	0			
Dock and Canal Companies' Debentures	5,000	0	0	110,000	0	0
City of London Corporation Bonds	4,000	0	0	16,000	0	0
Scottish Friendly Company's Securities	..			5,350	6	1
Agents' Balances	..			28,996	11	11
Outstanding Premiums	..			7,082	5	1
Outstanding Interest	..			16,004	12	10
Policy Stamps in hand	..			52	6	0
Bills Receivable	..			1,148	19	1
Cash in Banks	1,613	1	3	1,680	1	9
	£190,543	1	3	£969,362	3	5
				190,543	1	3
				£1,159,905	4	8

Balance Sheet of the Assurance Fund of the Company, shewing the Amount of Assets and Liabilities, and the Surplus Fund for division, at 25th March, 1870.

	£	s.	d.
ASSETS (as above)	£969,362	3	5

LIABILITIES.

	£	s.	d.
Present value of the Policies Assuring the Capital sum of £4,141,880	692,094	16	3
Present value of £230,851, Bonus declared at former divisions, now remaining	134,259	0	0
Present value of £707 11*s.* 5*d.* future annual reductions of Premium for life	5,865	0	0
Present value of £2,032 12*s.* 4*d.*, Annuities now payable	14,999	14	6
Present value of Deferred and Contingent Survivorship Annuities	1,169	0	0
Present value of the Scottish Friendly Company's Policies for the sum of £52,577	5,920	0	0
Claims admitted but not due till after the 25th March	32,478	10	0
	£886,786	0	0
Balance, being the Surplus Fund realised in the Five Years	82,576	2	8
	£969,362	3	5

The following particulars are extracted from the Returns under the Fifth and Sixth Schedules:—

[In the case of this office, we have not thought it desirable to republish the Summary and Valuation and the Valuation Balance Sheet, inasmuch as from the way in which they are arranged, they do not give any information as to the manner in which the surplus divided has been arrived at, and in fact the surplus shown in the Valuation Balance Sheet is widely different from that actually divided.—ED. *J. I. A.*]

2. The principles upon which the Valuation and Distribution of Profits among the Policy Holders are made are determined by the 99th clause of the Company's Deed of Settlement which directs "that the several "calculations shall be made by the Actuary of the Society for the time "being, or by some other person or persons to be approved of by the "Directors for the time being, at such rates of interest, and upon such "probabilities of the duration of human life, and upon such mathematical "principles for determining the respective contributions of the Assured to "the disposable funds of the Company as the Directors shall from time to "time think proper." And the same 99th clause further directs that "the disposable Profits of the Company shall be assigned or allotted to "them (the Assured on the participating scale), in shares directly propor-"tioned to their respective contributions to the disposable funds of the "Company."

8. Policies share at the first Division after they are effected, but the Bonus *vests* only if and when three years' Premiums have been paid.

Bonuses allotted to Policies in force for not exceeding Ten Years.

Age at Entry.	Policies in force for	
	5 Years.	10 Years.
20	4·8	5·0
30	4·8	5·0
40	4·8	5·1
50	4·8	5·2

Bonuses allotted to Policies in force for upwards of Ten Years.

Age at Entry.	Policies in force for						
	15 Years.	20 Years.	25 Years.	30 Years.	35 Years.	40 Years.	45 Years.
20	4	4	4	4	4	4	4
30	4	4	4	4	4	4	4
40	4	4	4	4	4	4	4
50	4	4	4	4	4	4	4

The amounts apportioned under the various modes in which the Bonus might be received were as follows:—

£57,113 in Reversionary Bonus.
1,958 in Reduction of Premiums.
830 in converting Whole Life Assurances into Endowment Assurances, payable at a fixed age, or at death.

£59,901

Consolidated Revenue Account of the Crown Life Assurance Company for Five Years, commencing 25th March, 1865, and ending 25th March, 1870.

	£	s.	d.		£	s.	d.
Amount of Funds on 25th March, 1865, the beginning of the period	1,008,590	15	6	Claims under Policies (after Deduction of Sums Re-assured)	459,168	8	5
Premiums (after Deduction of Re-Assurance Premiums)	527,334	14	9	Surrenders	24,775	14	7
Consideration for Annuities granted	1,672	9	4	Annuities	10,209	19	8
Interest and Dividends ..	219,111	17	4	Commission	27,073	3	0
Other Receipts—				Expenses of Management..	59,344	4	2
Fines from Proprietors for Non-Assurance ..	860	11	6	Dividends and Bonuses to Shareholders..........	67,237	16	2
Fines from Assured for Revival of Policies ..	53	4	2	Other Payments—			
Registration Fees......	119	0	0	Endowments	4,589	0	0
Balance of Profit and Loss Scottish Friendly Company	1,985	0	11	Claims Admitted, but not paid	32,478	10	0
	5,350	6	1	Dividends due, but not paid	1,144	16	3
Premiums, 1870, Outstanding	7,082	5	1	Amount of Funds on 25th March, 1870, the end of the period, as per First Schedule	1,126,281	18	5
Interest, 1870, Outstanding	16,004	12	10				
Bonuses to Shareholders, year 1865..........	24,138	13	2				
	1,812,303	10	8		1,812,303	10	8

The average Rate of Interest on the *Investments* of the Life Assurance Fund of the Company was, on 25th March, 1866, £4 9*s.* 6*d.*; 1867, £4 10*s.* 7*d.*; 1868, £4 11*s.* 2*d.*; 1869, £4 12*s.*; and 1870, £4 12*s.* 3*d.* per Cent. At the respective dates mentioned portions of the Fund were not invested; and if it be intended by the legislature that the rate is to be calculated on the *Gross* Fund, including Agents' Balances, Outstanding Premiums and Interest, and Interest accrued, &c., then the Interest on the *Invested* portion of the Fund was equal to the following rates on such *Gross* Fund, namely:—1866, £4 6*s.* 10*d.*; 1867, £4 5*s.* 5*d.*; 1868, £4 2*s.*; 1869, £4 6*s.* 5*d.*; and 1870, £4 7*s.* 5*d.*

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

On "*Extra Premium.*" *By* JAMES R. MACFADYEN, *of the Legal and General Life Assurance Society, Fellow of the Faculty of Actuaries (Scotland).*

[Read before the Institute, 25 March 1872.]

A PAPER on Extra Premium may treat of the subject in one or other of two ways. It may be, on the one hand, an attempt to deduce tables of mortality for the various classes of under-average risks occurring in a Life Office's transactions. Or, on the other hand, this part of the subject may be ignored and the paper be simply an enquiry into the effects resulting from the various methods employed in charging extra—an analysis of the actuarial points raised in consequence of a Society admitting not merely minimum risk lives, but others not eligible on similar terms. In short, the subject of Extra Premium may be considered either in respect to the amount to be imposed or in regard to the manner and effect of its imposition. The first question must be determined mainly by actual mortality observations—the second does not require in the same degree for its examination the light of experience. It is this latter question which I have taken up in the following paper; but as both branches of the subject are intimately connected, it may be well to make a few general remarks on the former also.

Inasmuch as the circumstances on which the rates of extra premium depend are infinitely varied, we can never have tables so complete as to be able to relegate every case that may arise in practice to a set of rates founded on observations of life in exactly similar circumstances. We can thus only administer a sort of rough justice—carry out the cardinal principle of life assurance—that the many must bear the burden of the one, and place every life that is insurable at all under such premium scale as shall most nearly represent its condition. In fact, the same thing is done in cases taken as minimum risks, and is the very essence of life insurance. The whole question then is: Where shall the line be drawn? How many tables of mortality shall be used? Were the selection of the assured against the Office done away with, one might suffice; and according to the force of this selection, will be the number of varying premium rates required. Anything tending to diminish this force will tend to simplify the classification of assured lives. As an illustration of this, let me take the following:—

It is at present the practice in many Offices to allow their policyholders free residence in any part of the world, provided the privilege be not exercised during a probationary period; this stipulation, together with a sufficiently broad foundation of general mortality, being considered adequate to check the force of selection against the Society. Similarly, any method acting as governor or regulator to the flow of assurances to an Office will obviate the necessity of a great variety of life premiums whether for disease or for climate, and will thus narrow the field of requisite enquiry as to the rates to be charged in under-average cases. With these remarks I shall leave the first branch of my subject and devote the remainder of this paper to the second form in which the question presents itself, namely, the actuarial results of the various methods now in use for charging extra premium.

Circumstances having an unfavourable tendency on the duration of human life may be divided into two great classes—the one containing those cases in which the age of the person affected has a more important bearing on the prejudicial circumstances, and the other, those cases in which less weight is attached to the time of life. I use the words "more" and "less" advisedly, because whatever be the nature of the circumstance the age must have some weight; but there are obviously certain kinds of risks in which this is more important than in others. Following the usual mode of classification—in the former section we see cases of con-

sumptive taint or tendency, rheumatic fever, and almost all the diseases, and also what may be called the "special circumstance" extras; in the latter, climate and occupation risks and a few maladies, such as hernia and gout, find a place. Let us now proceed to examine the first of these two sections.

The ordinary manner in which extra is charged in such cases is as follows:—

The Medical Examiner of the Office finding the proposer ineligible at the minimum rate, though not altogether uninsurable, recommends the acceptance of the risk with a certain number of years addition to the age; that, in fact, if the person be x years old, he be taken at the premium rate for, say age $(x+t)$. Let us assume that the life is accepted on these terms, and that P_{x+t} correctly represents the premium required to provide the benefit granted. Does it follow that on future occasions of dealing with the policy, the Office should treat it as if it were an ordinary assurance of entry age $(x+t)$? I believe that this is the method generally adopted. Let us now consider how far it is justifiable.

If we have any number of quantities a, b, c, d, e, &c., such that $a:b::c:d$ and $e:b::f:d$ and $g:b::h:d$, then we know that $a+e+g:b::c+f+h:d$; but the converse proposition is not true —that if $a+e+g:b::c+f+h:d$ it necessarily follows that

$$a:b::c:d$$
$$e:b::f:d$$
$$g:b::h:d$$

—the reason evidently being that we can completely vary the values of a, e, and g, and also c, f, and h, without affecting their sum, so that like ratios between the individual quantities do not necessarily follow, though they may possibly do so.

To apply this to my subject. A person aged x opening a policy of assurance by annual premiums has from some cause to pay extra to such an extent that he is charged the premium which would be the normal rate at age $(x+t)$. It does not follow that because his premium is the same as that of age $(x+t)$ his chance of death in any individual year will be the same as that of a person of real age $(x+t)$. Let dashes over the following symbols denote that mortality in accordance with extra is meant. Then (disregarding loading) since $P_{x+t}=P'_x$, we have $\frac{1}{1+a_{x+t}}-d=\frac{1}{1+a'_x}-d$, or $\frac{1}{1+a_{x+t}}=\frac{1}{1+a'_x}$, and therefore, $a_{x+t}=a'_x$; and similarly $A_{x+t}=A'_x$.

Now, by hypothesis,

$$\frac{(l_{x+t}-l_{x+t+1})v+(l_{x+t+1}-l_{x+t+2})v^2+\ \ldots\ .}{l_{x+t}(1+a_{x+t})}$$
$$=\frac{(l'_x-l'_{x+1})v+(l'_{x+1}-l'_{x+2})v^2+\ \ldots\ .}{l'_x(1+a'_x)}$$

i.e.,
$$\frac{(l_{x+t}-l_{x+t+1})v}{l_{x+t}(1+a_{x+t})}+\frac{(l_{x+t+1}-l_{x+t+2})v^2}{l_{x+t}(1+a_{x+t})}+\ \ldots\ .$$
$$=\frac{(l'_x-l'_{x+1})v}{l'_x(1+a_{x+t})}+\frac{(l'_{x+1}-l'_{x+2})v^2}{l'_x(1+a_{x+t})}+\ \ldots\ .$$

But as shown above, though the sum of the numerators of each side of this equation is to its common denominator as the sum of the numerators of the other side is to its denominator, it does not follow that each of the terms in one of these numerators is to its denominator as the corresponding term of the other numerator is to its denominator; and thus

$$\frac{(l_{x+t}-l_{x+t+1})v}{l_{x+t}(1+a_{x+t})}\text{ is not necessarily equal to }\frac{(l'_x-l_{x+1})v}{l'_x(1+a_{x+t})},$$

nor $\dfrac{(l_{x+t+1}-l_{x+t+2})v^2}{l_{x+t}(1+a_{x+t})}$ to $\dfrac{(l'_{x+1}-l_{x+2})v^2}{l'_x(1+a_{x+t})}$, nor &c.

Divide these ratios of possible inequality by $\dfrac{v}{1+a_{x+t}}, \dfrac{v^2}{1+a_{x+t}}$, &c., respectively, and we have

$$\frac{l_{x+t}-l_{x+t+1}}{l_{x+t}}\text{ not necessarily equal to }\frac{l'_x-l'_{x+1}}{l'_x},$$

nor $\dfrac{l_{x+t+1}-l_{x+t+2}}{l_{x+t}}$ to $\dfrac{l'_{x+1}-l'_{x+2}}{l'_x}$, nor &c.;

and generally, it does not follow that

$$\frac{l'_{x+n}-l'_{x+n+1}}{l'_x}=\frac{l_{x+t+n}-l_{x+t+n+1}}{l_{x+t}}.$$

So that if any life be charged an extra such that the premium paid is equivalent to that paid by a life t years older not subject to such extra, it does not follow that the chances of death in individual years after entry are the same in each case.

Similarly, though $\dfrac{av+ev^2+gv^3}{b}=\dfrac{cv+fv^2+hv^3}{d}$, it does not follow that $\dfrac{a+e+g}{b}=\dfrac{c+f+h}{d}$, since, so far as we know, the individual parts into which the numerators containing the powers of v may be split are quite arbitrary; and therefore, although the premiums to ages x and $(x+t)$ are the same, the expectations of life at entry as well as afterwards in each case may be different.

In the foregoing analysis it is not asserted that $P'_x = P_{x+t}$ for every value of v, but only for the one assumed in the calculation; so that the conclusions contrary to the preceding which would be arrived at from the rules concerning indeterminate coefficients do not apply to the case. Since, then, the numbers alive at successive ages after entry by the Table of Mortality followed by an under-average life, are not necessarily the same as the corresponding numbers t years older in a table of normal mortality, it follows that it may be wholly incorrect to assume that the chances of death of the person at extra risk will be the same as those of the older normal-risk individual, or that the worth of the reversionary sums will in each case be the same, or that the policy values will be identical; in short, that there is no warrant whatever for the practice in valuation or other calculations of treating under-average life as of the same worth as the corresponding older average life, even though it may have been identical at some former period.

I have said that it may be incorrect to assume this, but I go still further, and assert that it is *probably* incorrect, and that there is more likelihood that the mortality on the whole will be greater in individual years among lives at extra risk than among average lives t years older, even though at entry $P'_x = P_{x+t}$.

For since $P'_x = P_{x+t}$, as before $a'_x = a_{x+t}$, *i.e.*,

$$\frac{l'_{x+1}v}{l'_x} + \frac{l'_{x+2}v^2}{l'_x} + \ldots \frac{l'_{x+t+1}v^t}{l'_x} + \ldots = \frac{l_{x+t+1}}{l_{x+t}}v + \frac{l_{x+t+2}v^2}{l_{x+t}} + \ldots$$

Now some of the under-average lives may, and probably will, live as long as average lives of their age, or at least longer than is possible for any person of entry age $(x+t)$. Therefore the number of terms in the first member of the preceding equation is greater than the number in the second, and therefore, on the average, each term of the former is smaller; and as both series are diminishing, each term being less than the one preceding it, the earlier terms in each case have much greater weight in determining the value of the whole expression than the latter ones. It follows, therefore, that if the difference in the number of terms on each side is at all large, the earlier terms of the first member must on the whole be smaller than the corresponding terms of the second member, as otherwise the two series could not be equal. That is, though possibly not correct in individual cases, yet generally, $\frac{l'_{x+n}}{l'_x} < \frac{l_{x+t+n}}{l_{x+t}}$; and therefore

$$1 - \frac{l'_{x+n}}{l'_x} > 1 - \frac{l_{x+t+n}}{l_{x+t}}.$$

So that in under-average cases, where it was assumed at entry that a premium equal to that for an average life t years older will cover the risk, it is probable that in the earlier years of the assurance we shall find the anticipated deaths exceeded by the actual—meaning by the anticipated deaths those which would be found by taking the probabilities of an average life t years older dying within the period. This, however, is of course no proof that the extra is insufficient, as the latter years will make up the deficiency, and the few who live long will provide for those who die early. If, then, in calculating the expected mortality of an Office with a number of under-average risks on its books of comparatively recent entry, these lives are taken as equivalent to ordinary lives t years older, it follows that the expected mortality and claims will be under-estimated; and the greater the extras that have been charged, the greater on the whole will be the deficiency in the estimate.

Since, as we have seen, $\frac{l'_{x+1}}{l'_x}$ is probably less than $\frac{l_{x+t+1}}{l_{x+t}}$, it follows that $V'_{x|1}$ is probably less than $V_{x+t|1}$, and generally $V'_{x|n}$ than $V_{x+t|n}$.

For since

$$a_{x+t}=a'_x=\frac{l'_{x+1}}{l'_x}v(1+a'_{x+1}),$$

therefore

$$1+a'_{x+1}=\frac{l'_x}{l'_{x+1}v}a'_x.$$

Similarly,

$$1+a_{x+t+1}=\frac{l_{x+t}}{l_{x+t+1}v}a_{x+t}.$$

But if, as is probably the case, $\frac{l'_{x+1}}{l'_x}v<\frac{l_{x+t+1}}{l_{x+t}}v$, then it follows that $\frac{l'_x}{l'_{x+1}v}>\frac{l_{x+t}}{l_{x+t+1}v}$; and therefore, the values of a'_x and a_{x+t} being equal, $1+a'_{x+1}$ is greater than $1+a_{x+t+1}$, and therefore $1-\frac{1+a'_{x+1}}{1+a'_x}<1-\frac{1+a_{x+t+1}}{1+a_{x+t}}$, or $V'_{x|1}<V_{x+t|1}$, which inequality may be put in the form $\frac{1+a_{x+t+1}}{1+a'_{x+1}}<\frac{1+a_{x+t}}{1+a'_x}$. Similarly, on the further assumption of $\frac{l'_{x+2}}{l'_{x+1}}v<\frac{l_{x+t+2}}{l_{x+t+1}}v$, we have, for the inequality $V'_{x+1|1}>V_{x+t+1|1}$, the expression $\frac{1+a_{x+t+2}}{1+a'_{x+2}}<\frac{1+a_{x+t+1}}{1+a'_{x+1}}$, and therefore, *à fortiori*, $\frac{1+a_{x+t+2}}{1+a'_{x+2}}<\frac{1+a_{x+t}}{1+a'_x}$.

$$\therefore \ \frac{1+a_{x+t+2}}{1+a_{x+t}} < \frac{1+a'_{x+2}}{1+a'_{x}}. \quad \therefore \ 1-\frac{1+a'_{x+2}}{1+a'_{x}} < 1-\frac{1+a_{x+t+2}}{1+a_{x+t}},$$

i.e., $V'_{x|2} < V_{x+t|2}$, and in this manner on similar assumptions other policy values might be analysed—these assumptions having, as has already been pointed out, *à priori* evidence in their favour. So, then, if a Life Office in valuations of its policies, or in calculating the expected deaths among the number at risk, takes under-average lives as equivalent to normal lives of older age, the reserve required to be made to discharge its liabilities will probably be over-estimated, and the number of claims to be looked for under-estimated. It may be well to illustrate these rather important points by an example. Let us assume that there is an Insurance Society whose mortality rates among its average lives are in exact accordance with those of the Seventeen Offices' Experience Table, which table we shall consider to be the one it uses in calculations. Proposals for £100 each on lives aged respectively 25, 30, 35, 40, 45, 50, 55, 60 and 65 are made to the number of 100 at each age, and accepted by the Office, but being under-average cases at special rates of premium. Assume further, that the fixing of these premiums is not a mere guess on the part of the Society, but that it is aware that these persons will have a mortality among them similar to that prevalent, say among master mariners, as given in Neison's Vital Statistics, page 147, and fixes its rates accordingly. Suppose that at the end of one year from entry the Office wishes to compare its actual deaths with those expected, and throws in with the others these under-average cases, as normal lives of such entry ages as the premiums they pay would represent in the Experience Table; the following will be the result among these policyholders. I show also, for comparison, the effect of regarding the lives as ordinary cases of the entry ages above mentioned.

Real Entry Age.	Premiums paid, being those for Master Mariners' (Neison), represent in the Experience (17 Offices') Table an addition to Entry Age of about	"Experience" rated-up Age.
25	17 years................	42
30	14 years................	44
35	11 years................	46
40	9 years................	49
45	8 years................	53
50	6 years................	56
55	5 years................	60
60	5 years................	65
65	4 years................	69

Real Entry Age.	Expected Deaths among Policyholders according to "Experience" Rated Up.	Actual Deaths.	Expected Deaths among Policyholders according to "Experience" Real Ages.
25	1·089	1·971	·777
30	1·170	2·376	·842
35	1·284	2·343	·929
40	1·506	2·116	1·036
45	1·909	2·936	1·221
50	2·313	2·894	1·594
55	3·034	3·532	2·166
60	4·408	4·812	3·034
65	6·009	6·368	4·408
	22·722	29·348	16·007

By this method of treatment, then, the startling result will come out that the actual deaths exceed those allowed for by nearly 30 per-cent.

We shall next assume that the Society in valuing its liabilities, considers the foregoing policies as simply requiring the same reserve as that of average-risk contracts on lives aged 42, &c., respectively, at entry.

The following table will show the effect of this method of procedure. It will be observed that the policy values by the two different modes of calculation approximate as the age increases. This arises from the chance of death from normal causes increasing as the life becomes older, in a greater proportion than the chance from the special cause. For the sake of comparison, I also give the results by the Experience Table without any rating up of the age. In all the three methods of valuation, the interest has been taken as three per-cent.

By this mode of treatment it will be observed that in the illustrations chosen the true values are less than those standing for them by about 32 per-cent. The greater the extra added in each instance, the greater probably will be the error of substituting normal lives for under-average ones.

Another difficulty arising from this mode of considering extra, though not of like importance, may be stated. If the "rating up" be a very heavy one, and, as may well happen, the policyholder outlive the period between his assumed age and the oldest in the basis table, the actuary, when coming to make his calculations will be in the awkward predicament of having to deal with a person who by all rule ought to be dead, but obstinately persists

Real Entry Age.	Duration of Policy.	Value by Experience, "Rated Up."	Actual Value.	Value by Experience, Real Age.
25	1 year	1·812	·917	·946
	5 „	9·412	4·175	4·981
	10 „	19·445	8·131	10·626
		30·669	13·223	16·553
35	1 „	2·098	1·070	1·363
	5 „	10·731	6·358	7·161
	10 „	21·895	14·495	15·269
		34·724	21·923	23·793
45	1 „	2·633	1·576	2 033
	5 „	13·268	8·236	10·400
	10 „	26·559	19·712	21·277
		42·460	29·524	33·710
55	1 „	3·270	2·873	2·801
	5 „	16·141	14·107	14·077
	10 „	31·311	27·786	27·946
		50·722	44·766	44·824
65	1 „	4·095	3·752	3·732
	5 „	19·620	18·456	18 089
	10 „	36·941	35·260	34·472
		60·656	57·468	56·293
		219·231	166·904	175·173

in being alive. What is to be done in such a case? The life assured has escaped from the trammels of mortality tables, and, so far as appears from them, may, Tithônus like, flourish for ever —a perpetuity in favour of the Office. No doubt the same thing might arise even were no "rating up" made, but a heavy extra would render it a much more likely occurrence. I do not urge this objection, however, as of great moment in itself, but bring it forward mainly because it contains a sort of *reductio ad absurdum* proof of the error of the "rating up" method of treatment.

We have seen then that even if at entry the rate paid by an under-average life coincides with that charged on an older healthy life, it does not follow that in after years the policy can be considered as identical with that of a policy of similar duration on the older life, and that in fact, so far from the presumption being in favour of their being the same, the reverse is the case. How then can we value assurances at extra risk, or compare at any given moment the mortality experienced with that assumed in fixing the premium? By the present system of dealing with each instance that arises, not as one of many, but as an individual

and isolated assurance, there is no way whatever by which we can do so. In order to make calculations concerning a policy, we must have a mortality table, or what is the same thing, a complete set of the premiums for ages older than x that would be charged persons in a similar position as to circumstances affecting health. With this, we can of course arrive at the values of annuities and reversions, the death rate provided against, and, in short, all the requisite tables for our objects. Without this we can do nothing. At present, after the Medical Adviser has estimated the number of years addition to be charged, the Actuary, in order to deal with the policy, sometimes makes the assumption as to "rating up" we have already referred to. In doing so, he is practically undertaking the duty of the Medical Officer of the Society, as he is simply settling the premiums to be charged for similar risks of higher entry-ages, and, like amateurs in general when undertaking duties requiring special training, he is, as I have already shown, probably wrong in his medical opinions. The only defensible method of treating such assurances is to have a variety of complete sets of premium rates, to one or other of which, according to the intensity of the health-affecting circumstances, the Medical Adviser of the Company can relegate each case that arises. These sets, as I have said in a former part of this paper, need not be numerous; just enough to prevent selection against the Office to a dangerous extent will suffice. Let us, if we may, form these rates from the experience of the principal classes of extras, but empirically if we must. In saying this, I am, of course, looking at the question not so much in regard to sufficiency of premium as to consistency. Like the Legate in Browning's poem, I am not now objecting to the doctors calling two and two five, if they will only go on and call four and four ten. If, as we know they do at present, they can evolve from their own consciousness a single rate, let them go further and evolve a whole table. It demands no further stretch of the imaginative faculties, and it will at least let actuaries know how they stand with regard to extra premium risks.

I shall now proceed to examine the second form in which extra premium occurs, viz., a fixed amount of addition quite irrespective of age. As we here have a complete set of rates to which every case is relegated, the great difficulty in the method previously examined is of course avoided. As I have formerly said, however, though the age of the life assured may not be of the same importance in extra risks of this class it must have a

certain weight. Mr. Makeham, in his ingenious paper on this branch of my subject (*Journal of the Institute of Actuaries,* vol. xiv, page 159), points out that the imposition of a fixed extra is, to quote his own words, "evidently a sort of rule of thumb way of "giving effect to the notion that the extra risk acts with equal "force at all ages;" and by a mathematical analysis of this principle he arrives at the very convenient result, that an increase in the rate of interest by the required amount of extra mortality force, will furnish us with all the details for making calculations concerning the policy.

As Mr. Makeham's paper is but a sketch, I may be pardoned for trenching on the field he has opened up by examining somewhat in detail various interesting points connected with his method of treating extra.

Supposing Mr. Makeham's principle of dealing with such risks to be the object aimed at in the usual practice of a constant added premium irrespective of the period of life, if we assume the rate now charged be correct at any given age, at higher ages it will be increasingly too much, and the error will be a maximum at the oldest age in the tables. For, employing dashes as formerly to signify mortality including extra, by the method now in use of charging the same amount of addition whatever be the age, we have

$$P'_x - P_x = P'_{x+n} - P_{x+n},$$

i.e.,

$$\frac{1}{1+a'_x} - \frac{1}{1+a_x} = \frac{1}{1+a'_{x+n}} - \frac{1}{1+a_{x+n}}.$$

Now, by Mr. Makeham's principle we may consider the dashes as representing not an increased rate of mortality, but an increased interest rate.

Let Δ_x be the difference between the annuities at age x by the lower and higher interest rates; and Δ_{x+n} similarly at age $(x+n)$, that is, $\Delta_x = a_x - a'_x$ and $\Delta_{x+n} = a_{x+n} - a'_{x+n}$.

Then we have by the ordinary mode of charging extra

$$\frac{1}{1+a_x-\Delta_x} - \frac{1}{1+a_x} = \frac{1}{1+a_{x+n}-\Delta_{x+n}} - \frac{1}{1+a_{x+n}}.$$

Now, as the age increases the differences between annuities at different rates of interest tend to vanish, *i.e.,* generally $\Delta_x > \Delta_{x+n}$. For making $\kappa v = v'$ we have

$$a_x - a'_x = \Delta_x = \frac{l_{x+1}v(1-\kappa) + l_{x+2}v^2(1-\kappa^2) + \ldots\, l_{x+n+1}v^{n+1}(1-\kappa^{n+1}) + \ldots}{l_x}$$

and

$$a_{x+n} - a'_{x+n} = \Delta_{x+n} = \frac{l_{x+n+1}v(1-\kappa) + l_{x+n+2}v^2(1-\kappa^2) + \quad \ldots\,.}{l_{x+n}}$$

But $\frac{l_{x+1}}{l_x}$ is generally greater than $\frac{l_{x+n+1}}{l_{x+n}}$, and so with the other terms, of which there are also a greater number in Δ_x, so that it may be said that $\Delta_x > \Delta_{x+n}$.

Δ_{x+n} then constantly diminishes with the increase of n till at the extremity of the table it reaches the limit 0, so that as the age increases the difference between the premiums with extra and without extra should approach the limit 0, while by the system in use at present the added premium remains always equal to the extra demanded at age x, so if the premium be correct for that age by our usual practice, there is an increasing overcharge on older lives. The instances given in Mr. Makeham's letter practically illustrate this analysis, as the extras in them form a constantly diminishing series.

In the paper already mentioned, the question is examined as to whether, in extra risks for climate and the like, it is legitimate, as is now usually done, to altogether remit the addition to the premium on the return of the life assured to this country, and the conclusion is arrived at that the ordinary practice is erroneous. As the matter is of considerable practical importance, I consider no apology necessary for introducing an analysis of the matter varying somewhat from that of Mr. Makeham.

Fully accepting his axiom that the correct principle for finding the premium that should be charged on return, is that the liability of the office must be left unchanged by the alteration in the contract, I shall now proceed to examine the question. Let one dash over any symbol signify that the table of mortality on which it is based is that which the life obeys while the cause for which extra was imposed is in action, while two dashes similarly represent the rates of mortality that will prevail among lives once subject to such extra risk, but which are no longer so. Then, neglecting loadings, P is such that the value of the policy at the date (age $x+n$) when extra ceases to act, applied to purchase an annuity for the remainder of life, and the result subtracted from the premium that would now have to be paid for a new assurance, will give the future annual payment.

That is, $$P''_{x+n} - \frac{V'_{x|n}}{1+a''_{x+n}} = P.$$

If we assume that those lives who have been subject to the increased risk, but are so no longer, will obey the same mortality table as those who have never experienced the extra risk

at all, the preceding formula will become

$$P_{x+n} - \frac{V'_{x|n}}{1+a_{x+n}} = P.$$

which is another form of Mr. Makeham's expression.

He then in his analysis takes it for granted that lives previously subject to increased risk, but not so now, will be in exactly the same state as to health as if they had never undergone any special danger whatever, an assumption which seems to me very much open to question. The correctness or incorrectness of it, however, belongs to the other branch of my subject, and it is sufficient for me here to point out that this assumption is made. I have said that the ordinary practice is to take off the extra altogether, that is (taking Mr. Makeham's expression for the future premium), to assume that

$$P = P_x = P_{x+n} - \frac{V'_{x|n}}{1+a_{x+n}}.$$

But we know that $P_{x+n} = \dfrac{V_{x|n}}{1+a_{x+n}} + P_x$.

Substituting this, we have it correct to remit extra altogether, if

$$\frac{V_{x|n}}{1+a_{x+n}} + P_x - \frac{V'_{x|n}}{1+a_{x+n}} = P_x,$$

i.e., if
$$\frac{V_{x|n}}{1+a_{x+n}} = \frac{V'_{x|n}}{1+a_{x+n}},$$

i.e., if
$$V_{x|n} = V'_{x|n}.$$

And since, applying Mr. Makeham's method of treating extra, $V'_{x|n}$ means simply the value of a similar policy to $V_{x|n}$ by the same table of mortality, but a higher rate of interest, it follows that it is only legitimate to reduce the premium to the home scale, if the value of a policy at a higher rate of interest is the same as that at a lower one, a thing which we practically* know cannot be the

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* NOTE.—I use the word "practically" because I have not seen and have not myself been able to arrive at a theoretical proof which is entirely satisfactory. The following shows that in any given case it is practically certain that the value of a policy by a higher rate of interest must always be less than the value by a lower.

Let $V_{x|1}$ be the value of a policy opened at age x, the present age being $(x+1)$, when v is the present value of £1 discounted for one year, and let $V'_{x|1}$ be the corresponding value by the same table of mortality when r is the discounted value of £1 for one year, v being assumed greater than r. The question then is, which is greater—$V_{x|1}$ or $V'_{x|1}$?

case. Mr. Makeham in his paper arrives at the same conclusion, that the reduction to the home rate is unwarranted, by a proof founded on special calculations.

Since in the foregoing analysis the greater the extra force of mortality, the greater will be the difference (through interest) between $V_{x|n}$ and $V'_{x|n}$, it follows that the greater the added premium imposed, the less near the truth is the custom of taking off extra payment when extra risk ceases. The practice, however, may be defended on the ground that the charge made during the period when the special risk is incurred exceeds the amount actually necessary by such a sum as will, when accumulated, provide for the reduction of the premium to minimum rates on the special risk ceasing. This can only be determined by an enquiry as to whether the premium imposed actually allows of this, —a matter belonging to the other branch of my subject.

Assuming that after the cause for which extra payment was charged ceases to act, the mortality will be the same as among lives never subject to the increased risk, we have already seen that the preceding expression represents the future premium exigible, and that by using this equation the reserve the Office has to maintain against the policy will be unchanged. If, however, the premium be reduced to P_x (the rate that would have been charged

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$$1-\frac{1+a_{x+1}}{1+a_x} >< 1-\frac{1+a'_{x+1}}{1+a'_x}$$

i.e.,

$$\frac{1+a'_{x+1}}{1+a'_x} >< \frac{1+a_{x+1}}{1+a_x};$$

but $1+a'_{x+1}=\frac{l_x}{l_{x+1}r}a'_x$, and $1+a_{x+1}=\frac{l_x}{l_{x+1}v}a_x$. Substituting these values and dividing by $\frac{l_x}{l_{x+1}}$, we have

$$\frac{a'_x}{r(1+a'_x)} >< \frac{a_x}{v(1+a_x)}$$

or, multiplying out and transposing,

$$a'_x a_x(v-r) >< ra_x - va'_x$$

that is

$$(v-r)a'_x\frac{l_{x+1}v+l_{x+2}v^2+\dots}{l_x} >< \frac{l_{x+1}rv+l_{x+2}rv^2+\dots}{l_x}$$
$$-\frac{l_{x+1}rv+l_{x+2}r^2v+\dots}{l_x}$$

that is

$$(v-r)a'_x\frac{l_{x+1}v+l_{x+2}v^2+\dots}{l_x} >< (v-r)\,\overline{l_{x+2}rv+l_{x+3}(v^2r+r^2v)+\dots}$$

Divide by $(v-r)$ and multiply by l_x and we have

$$(l_{x+1}v+l_{x+2}v^2+\dots)a'_x >< l_{x+2}vr+l_{x+3}(v^2r+r^2v)+\dots$$

at entry for an assurance without extra risk), unless $V'_{x|n}=V_{x|n}$, the liability of the Office is at once altered, and our calculation of the necessary reserve upset. And if, as I have shown to be probably the case, the reduction to P_x is greater than is warrantable, then not only is the amount held by the Society to discharge its contract altered, but changed in the direction contrary to the interests of the Office. It will require a greater sum to discharge its obligations, so that unless in the unlikely case of $V'_{x|n}$ being equal to $V_{x|n}$, in any Office wholly taking off extra on the removal of special risk, the policyholder has at any time the option of upsetting the calculations of the Company as to the reserve it ought to keep, and either diminishing, or much more probably increasing, the liability of the Office in a manner not provided for in its periodical valuations. No doubt it is possible, by substituting P_x for the future premium in the foregoing equations, to find such a value for $V'_{x|n}$ as will permit of the reduction being given. But here, again, we labour under the disadvantage that if this be done in all contracts having this option, the liability under policies not then

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Multiplying out and arranging we have

$$\left.\begin{array}{l} vl_{x+1}a'_x \\ +v^2l_{x+2}a'_x \\ +v^3l_{x+3}a'_x \\ +\;.\;.\;.\;.\;. \\ +v^{\omega-2}l_{x+\omega-2}a'_x \\ +v^{\omega-1}l_{x+\omega-1}a'_x \end{array}\right\} >< \left\{\begin{array}{ll} v(l_{x+2}r+l_{x+3}r^2+\;.\;.\;.\;.\;) & (=vl_{x+1}a'_{x+1}) \\ +v^2(l_{x+3}r+l_{x+4}r^2+\;.\;.\;.\;) & (=v^2l_{x+2}a'_{x+2}) \\ +v^3(l_{x+4}r+l_{x+5}r^2+\;.\;.\;.\;) & (=v^3l_{x+3}a'_{x+3}) \\ +\;.\;.\;.\;.\;.\;. & (=\;.\;.\;.\;) \\ +v^{\omega-2}l_{x+\omega-1}r & (=v^{\omega-2}l_{x+\omega-2}a'_{\omega-2}) \end{array}\right.$$

From the foregoing expression then it will be seen that it is practically certain that $V_{x|1}$ must exceed $V'_{x|1}$. For though in an individual instance the annuity at an older age may be greater than that at a younger, yet this excess can never be so much, even when all possible cases of it are taken into account, as to counterbalance the facts that in the left member of the foregoing inequality we have one more term than in the right member, and that also a'_x is constant in the left, while in the right it is replaced by the diminishing series of annuities a'_{x+1}, a'_{x+2}, &c. Since then we know that $V_{x|1}$ must be greater than $V'_{x|1}$,

therefore $$\frac{1+a'_{x+1}}{1+a'_x}>\frac{1+a_{x+1}}{1+a_x}$$

i.e., $$\frac{1+a'_{x+1}}{1+a_{x+1}}>\frac{1+a'_x}{1+a_x}$$

Substituting $(x+1)$ for x, we have

$$\frac{1+a'_{x+2}}{1+a_{x+2}}>\frac{1+a'_{x+1}}{1+a_{x+1}},\text{ and \textit{à fortiori} }\frac{1+a'_{x+2}}{1+a_{x+2}}>\frac{1+a'_x}{1+a_x}\;\therefore\;V_{x|2}>V'_{x|2},$$

and so on with other ages. Therefore it may be asserted that $V_{x|n}>V'_{x|n}$.

exercising this privilege will be erroneously estimated; and, in fact, it is only by importing into our calculations the additional element of the chance of the option being exercised that we can have the values of such contracts correctly arrived at. It appears, then, that as in the other form of extra, where the life assured was "rated up," so in this form, where this option is granted, the ordinary method of valuation leaves it quite unsettled what amount must be reserved to meet the liability incurred: in the former, from the want of mortality tables; in the latter, from the option of reduction to normal entry rate;—a most unsatisfactory state of things, and a matter of no small moment to Companies issuing such policies in considerable numbers.

I have now examined some of the principal phases presented by the two modes of imposing extra premium, but though it will make somewhat longer a paper already sufficiently lengthy, it will be well, before concluding, to glance at a very interesting method of charging extra in use by some Offices.

In a few Societies, the life assured, if accepted at an annual extra of f, is at liberty to pay the ordinary premium, it being stipulated that in the event of death occurring in the first year, a deduction of ef will be made from the sum assured (e being the average after-lifetime of an *ordinary* life, aged x), and in consecutive years thereafter $(e-1)f$, $(e-2)f$, &c., are respectively deducted, according to the number of years from entry at which the claim arises. Is this mode of treating extra, or the ordinary one of charging f, year after year, likely to be the more profitable to the Office?

Let, as before, dashes signify mortality at under-average rates. By this method the life assured gives the Society an assurance of ef, diminishing f per annum.

This equals

$$f\frac{l'_x-l'_{x+1}}{l'_x}v+f\frac{l'_x-l'_{x+2}}{l'_x}v^2+\ \ldots\ f\frac{l'_x-l'_{x+e-1}}{l'_x}v^{e-1}+f\frac{l'_x-l'_{x+e}}{l'_x}v^e$$

or $$f\left(1-\frac{l'_{x+1}}{l'_x}\right)v+f\left(1-\frac{l'_{x+2}}{l'_x}\right)v^2+\ \ldots$$

$$+\ f\left(1-\frac{l'_{x+e-1}}{l'_x}\right)v^{e-1}+f\left(1-\frac{l'_{x+e}}{l'_x}\right)v^e;$$

while by the more usual method of charging extra, the life assured gives the Society an annuity-due of f, that is

$$f+f\frac{l'_{x+1}}{l'_x}v+f\frac{l'_{x+2}}{l'_x}v^2+\ \ldots\ f\frac{l'_{x+e-1}}{l'_x}v^{e-1}+f\frac{l'_{x+e}}{l'_x}v^e+f_{e|}a'_x.$$

Divide each by f, and we have the two series—

$$\left(1-\frac{l'_{x+1}}{l'_x}\right)v+\left(1-\frac{l'_{x+2}}{l'_x}\right)v^2+\ \ldots\ \left(1-\frac{l'_{x+e-1}}{l'_x}\right)v^{e-1}+\left(1-\frac{l'_{x+e}}{l'_x}\right)v^e$$

and

$$1+\frac{l'_{x+1}}{l'_x}v+\frac{l'_{x+2}}{l'_x}v^2+\ \ldots\ \frac{l'_{x+e-1}}{l'_x}v^{e-1}+\frac{l'_{x+e}}{l'_x}v^e+{}_{e|}a'_x.$$

Multiplying out the first series, and transposing from the one to the other, we have the expression for the ordinary form of charging extra,

$$1+2\left\{\frac{l'_{x+1}}{l'_x}v+\frac{l'_{x+2}}{l'_x}v^2+\ \ldots\ldots\ \frac{l'_{x+e-1}}{l'_x}v^{e-1}+\frac{l'_{x+e}}{l'_x}v^e\right\}+{}_{e|}a'_x,$$

and the expression for the method we are now considering,

$$v+v^2+\ \ldots\ldots\ v^{e-1}+v^e.$$

That is, the ordinary form gives an under-average life annuity due + an annuity on an under-average life for e years, and the other gives an annuity certain for a number of years equal to the expectation of an average life. Which is greater?

If the probabilities of an under-average life were those of an average one, then, as in e years, since at very few ages at which assurances take place does $\frac{l_{x+e}}{l_x}$ fall under $\frac{1}{2}$, it follows that $\frac{l_{x+1}}{l_x}$, $\frac{l_{x+2}}{l_x}$, &c., would form a diminishing series, of which the value of the smallest term was not less than $\frac{1}{2}$, and we should thus have

$$2\left(\frac{l_{x+1}}{l_x}v+\frac{l_{x+2}}{l_x}v^2+\ \ldots\ldots\ (\tfrac{1}{2}+\delta)v^e\right)$$

greater than $v+v^2+\ \ldots\ldots\ v^e$, and, *à fortiori*, the whole of the first series greater than the second; therefore, to a life which has been erroneously classed as under-average, the new method of charging extra is much the more advantageous.

But in the case when the life really involves extra risk, since $\frac{l'_{x+e}}{l'_x}$ is less than $\frac{l_{x+e}}{l_x}$, it may possibly happen that the chance of the under-average life attaining the expectation age of an average life is less than $\frac{1}{2}$, and thus we cannot say whether the new or the old method will be the more advantageous for the Office, as it altogether depends on the probabilities of invalid life reaching the various ages.

Were even the number alive at the age $\left(x+\frac{e}{2}\right)$ as great as half those at entry age, a supposition probable enough even in assurances at extra risk, the first series would be

$$1+2\{(\tfrac{1}{2}+a)v+(\tfrac{1}{2}+a-\beta)v^2+\,..\,\tfrac{1}{2}v^{\frac{e}{2}}+(\tfrac{1}{2}-\psi)^{\frac{e}{2}+1}+\,..\,(\tfrac{1}{2}-\theta)v^e\}+{}_e a'_x$$

and the second

$$v+v^2+\,\ldots\,v^{\frac{e}{2}}+v^{\frac{e}{2}+1}+\,\ldots\,v^e$$

of which, since the coefficients of the powers of v up to $v^{\frac{e}{2}}$ in the bracket are greater than the corresponding coefficients in the latter series, it is probable, taking also into account the term ${}_e a'_x$, that the first expression as a whole is greater than the second. That is, even when in a number of years, equal only to half the expectation of an average life, there are no more than $\frac{l'_x}{2}$ survivors of the l'_x living at age x, it would be more advantageous to the Office to charge extra during life than by the method we are now considering. When the probabilities of death become very great in the first few years, this way of meeting the extra risk may give more favourable results to the Society than the more usual one. In such a case, the age which the probability of reaching is less than $\frac{1}{2}$ must be very much less than that which represents the average age at death of ordinary assurers. It is very unlikely, however, that in many cases the chances of death in the first few years of assurance will be so great as to cause the value of an annuity certain for a period equal to the expectation of an average life, to exceed twice the value of an annuity of the same number of terms on an invalid life by $1+$the value of the deferred annuity; and so, though it is impossible to speak with absolute certainty, it is exceedingly probable that the ordinary system of charging extra during life is more profitable for the Office than the method now being considered, and since in each case the assumption is made that only the value of the extra risk incurred is charged, it is likely that this new system will fail to repay the Society for the increased liability undertaken.

In my treatment of the subject of extra premium, I have altogether neglected the question of loading. I have done so, partly because its introduction would very much complicate the elucidation of these features I have more particularly desired to exhibit; and partly because I believe that the subject of the loading on the risk premium ought to be considered in conjunction with the method adopted in dividing surplus—a field of enquiry much too wide for the scope of the present paper.

I have now examined some of the leading phases in which extra premium presents itself. Whatever may be thought of my analysis, I shall not have written altogether fruitlessly if I succeed in calling the attention of the Institute to the subject. When other branches of the science of life insurance are rapidly advancing, this seems in very much the same position as it was when the actuarial profession was yet in its infancy—anything but a satisfactory state of matters, when we consider the practical importance of the subject, and how intimately connected it may be with the solvency of a Life Office.

The Influence of Occupation upon Health, as shown by the Mortality experienced. By Francis G. P. Neison, F.S.S.

[Read before the Institute, 29th April 1872.]

THE influence that occupation has upon health is a subject that has hitherto attracted but little attention, for though in the works of one or two writers on vital statistics it has been slightly alluded to, no work has yet appeared which, while founded on direct observation, has been devoted to the treatment of this subject.

Perhaps also it was not thought that the results would be such as to much affect any of those questions in which the duration of life enters into account, and which concern so vitally the welfare of the numerous Life Offices and Benefit Societies of this kingdom; and that, therefore, the investigation would not repay its labour. When, however, the results now submitted are considered, the importance of the question will be fully revealed, and it will be perceived that the influence of occupation is one that essentially concerns all those Societies in which many of like or different trades are associated together for the purpose of securing contingent benefits.

A brief description of the data on which the results presented are founded, may be now given.

The Census of 1861 afforded a ready means of ascertaining the number living of any given occupation in that year; and the Supplement to the 25th Annual Report of the Registrar-General gave the occupations of all males that died in the years 1860, 1861.

There was thus the means of ascertaining the mortality experienced by various occupations in those years; and Dr. Farr, the able Assistant of the Registrar-General, availed himself of the opportunity, and gave, in the Supplement above referred to, some interesting results for one or two occupations.

On a careful examination of these records, it was found that though for many trades most accurate results could be arrived at, yet there were others for which, on account of their indefinite character, no satisfactory conclusions could be obtained. The principal thing to be borne in mind, was that only such occupations should be considered as, on account of their well-defined character, would preclude any chance of the living having been entered at the Census under one occupation, and the deaths in the Registrar's Report under another, or *vice versâ*. For instance, a butcher's is a well defined trade, which one can hardly imagine being conglomerated with any other pursuit; but with many occupations it would be impossible to provide against persons following them being returned at one time under one, and at another, under a different occupation; so that in all trades and occupations in which there was any such tendency, any trustworthy method of determining their mortality was precluded. Unfortunately, of this description was one of the most important sections of the working classes, namely, those included under the general term of "Labourers." In the Census they were entered under various heads, such as "Agricultural Labourers," "Farm Labourers," "Road" and "General" Labourers, &c.; whereas, in the death Registers, numbers are returned under the general heading of "Labourers" alone; and only some under the respective headings in which they were classed at the time of taking the Census. Throughout the paper the classification of the different occupations as presented in the Census has been, as far as possible, retained, so as to facilitate reference.

The objections that on first thoughts would occur as being incidental to any conclusions based on the Census, and the Death Registers, arising from the geometrical progression of the population, inaccuracies in the statement of ages, etc., in reality exercise but an immaterial influence, and one which does not practically much affect the accuracy of the results, as Dr. Farr has pointed out in his introduction to the English Life Table.

The period of life taken for the illustration of the influence of occupation upon health will be the 40 years intervening between 25 and 65 years of age. The reason for fixing on this period of life was, that though the Census gave the ages of the living for quinquennial ages, yet in the Supplement to the 25th Annual Report the ages at death were shown for only decennial periods. However, any influence that occupation had upon the mortality experienced would have ample scope for displaying itself in the 40

years under observation; and as it certainly embraces the best part of the working years of a man's life, it has perhaps its advantages over the only alternative, namely 15 to 75, which, while including in most occupations quite novices in their art, would also have taken into account a time of life when great bodily or mental activity must have, in a great degree, ceased, and the influence of occupation be materially decreased.

As somewhat of a test of the accuracy of the basis on which the present results are founded, they may be compared with those of a totally independent investigation which I have recently completed, but which has not yet been published.

The latter enquiry was based on the experience of members of Friendly Societies, and embraced altogether 1,147,243 years of life. There are some points, however, in which the two datas would differ. In one case, being all members of Benefit Societies, they would have had to pass a medical examination, and might therefore in some degree be regarded as select lives; though it may fairly be questioned whether a medical examination would render the lives of Members of Friendly Societies much better than those of the same occupation in the country generally, as it is difficult to comprehend in what way such an examination could provide against the influence attributed to occupation. At any rate, it could not exert much effect beyond the first few years, when any disease engendered by trade or occupation would begin to show itself; but which symptoms, if they had been in existence at the time of the medical examination, might have precluded the person entering the Society. Even though the effect of this selection may, after the lapse of a year or two, yield to the greater influence of occupation, it should be remembered that at no time, in any occupation, amongst members of Friendly Societies, would there be many addicted much to irregular habits; as if they were not expelled on that account, the improvidence which such a course of life soon begets would render their membership in the Society a mere matter of time, as, sooner or later, would the requisite monthly or weekly subscription cease. Therefore, as not including any of the dissolute or improvident, to some extent the mortality might naturally be expected to be somewhat less than that of members of the same occupation in England and Wales generally.

The mortality for the period of life, 25 to 65, is appended for a few occupations, as demonstrating how wonderfully the two investigations corroborate one another.

Occupation.	Mortality per 1,000 living.	
	Present Paper.	Friendly Societies.
Gardeners	10·4	10·6
Carpenters and Joiners..	12·7	12·4
Shoemakers	14·5	13·4
Stone Masons	17·6	16·8
Butchers	17·4	17·9
Innkeepers and Publicans	25·0	21·4

Perhaps no better example could be given of the great influence of occupation upon health than the foregoing facts; and it is all the more satisfactory, considering the wide bounds within which the mortality might fall, to perceive the remarkable coincidence of the two results.

In regarding the facts that will presently be given, it should be remembered, that whereas, in the standard mortality tables at present in use, taking for instance those two which may be said to fairly represent the extremes, namely, Dr. Farr's English Life Table No. 3, and Mr. Neison's Friendly Society Experience Table, the differences of the mortality for the 40 years of life, 25 to 65, would be confined between the two ratios of 16·7 and 13·0 per 1,000 living; yet so great is the influence of occupation, that, whilst for the same period of life the mortality amongst Gardeners in domestic service would be but at the rate of not quite 8 per 1,000, that amongst Inn and Hotelkeepers would be 26·8 per 1,000. Thus, so considerable is the influence of occupation, that the mortality in one avocation exceeds that in another by as much as 239 per-cent.

We hear a great deal of the sanitary conditions of our towns and cities, and the many improvements needed in them; but surely a little attention paid to an investigation into the causes of this great difference in the health of our various trades, with a view to the removal thereof, would amply repay its labour, and without doubt tend, more than anything else, to the better health of our towns and cities.

In all the tables the mortality is given for three distinct periods of life. First, for the 40 years 25 to 65; so as to exhibit the effect of the occupation when spread over the whole period under consideration; and secondly, the same sub-divided into two parts, so as to ascertain in what degree the young and old ages are respectively affected. Finally, a column will be appended, exhibiting the mean mortality for those two periods, so as to counteract any dis-

turbing element in the form of a preponderance of either old or young lives in any special occupation when compared with another. This arrangement, it is contended, will amply show any influence of occupation, and be preferable to dividing the period into more portions, in which case the labour of following out the results would, it is thought, be too tedious.

It may here be stated that the mortality for the whole of England and Wales during the years under review, namely, 1860 and 1861, was, for the 40 years of life, 25 to 65, 1·50 per-cent; and during each of the 20 years of life, 25 to 45, 45 to 65, respectively, 1·05 and 2·30 per-cent. If these few figures are carefully remembered, the trouble will be amply repaid by the ready means thus afforded of comparing the mortality of any one class with that of the country generally.

Reference will be made in one or two parts of this paper to the Third Report of the Medical Officer of the Privy Council. Dr. Greenhow's investigations acquire more particular importance from the circumstance that his observations, having been conducted in the year 1860, refer to the identical lives composing the basis of the present results.

As a paper of this description must necessarily be somewhat circumscribed, instead of regarding a number of trades or avocations separately, certain combinations of occupation will be submitted.

The Influence of Underground Occupations upon Health.

The effect that mining has upon the health of those engaged in it, is the object of the first combination that will be presented, and the accompanying table A displays the influence of various underground occupations upon health. *(See Table A.)*

Iron Miners, it will be perceived, are the most healthy of all Miners, and what is more remarkable, are subject to a mortality only just on a par with that of the country generally.

Coal Miners come next on the list; their mortality being slightly in excess of that of Iron Miners. That these two classes, comprising together a body of 267,239 persons, should be so healthy, notwithstanding the many injurious influences, and, especially in the case of Coal Miners, accidents, to which they are exposed, will be regarded as exceedingly satisfactory.

Whilst treating of Coal Miners, it may be observed that the quarter of the kingdom in which the mining operations are carried on has very materially to do with the health of the Miners.

Coal Mining, as carried on in the North of England, and the same as effected for instance in Staffordshire, are essentially different. Whilst the Mines in the one belong to wealthy proprietors, and have every artificial and mechanical improvement, both as relates to their ventilation, and mode of working, those in Staffordshire are, I believe, owned only by small capitalists, worked moreover by a different process, and much inferior in ventilation and other matters affecting the health of the Miners. In fact, the health of the Coal Miner in the midland counties is not nearly on a par with that of his northern fellow-labourer. In Northumberland and Durham the health of the Coal Miner is at its maximum, and in the South Wales District at its minimum.

Again, the deaths from violence in the South Wales and Staffordshire Districts are double, and at the younger ages even treble, the rate prevalent in the north of England. The accidental deaths arising from explosions and choke-damp do not form nearly so high a percentage of the total deaths from accidents as is generally thought; the principal cause of accident being from the want of proper supports to prevent the coal falling and crushing the Miners Doubtless this is in many cases to be attributed to negligence on the part of the Miners themselves, though in some Mines it is to be feared "that the men, to save labour, and the agents to save cost, "omit to put up the proper supports."—(*Vide* Royal Commission on Mines).

I would refer all interested in this question to the elaborate inquiry into the health of Coal Miners that was instituted a few years ago by the Royal Commission on Mines. An invaluable amount of information, and statistics on the subject, were then obtained.

The difference between the mortality of Iron and Coal Miners and the three classes that follow, is very great, and deserves marked attention, inasmuch as it intimates that there is some ruling influence predominating in these three classes, which Iron and Coal Miners are, more or less, free from.

Tin Miners appear to experience a high mortality at the older ages, it being just over three per-cent; whilst, as regards Lead Miners, the mortality for the 40 years, 25 to 65, was over two per-cent, and at the older ages also very high, namely 3·6 per-cent. The most unhealthy of all, however, were Copper Miners, the death-rate in this occupation being for the period of life 25 to 65, at the rate of 2·4 per-cent, and at the older ages but a fraction short of five per-cent.

A correct determination of the cause of this greatly increased mortality amongst Tin, Lead, and Copper Miners, is of the greatest importance. May it not be possible that the metal itself has some material influence, inasmuch as it can be demonstrated that all engaged in occupations connected with the latter three metals have uniformly a higher mortality than those engaged in trades or occupations in connection with iron?

From some inquiries made with reference to the tendency of Tin, Lead, and Copper Mining to produce diseases of the Lungs, the results of which investigation, conducted by Dr. Greenhow, are contained in the Third Report of the Medical Officer of the Privy Council, it was ascertained that these Miners are peculiarly prone to pulmonary diseases.

First, as regards the Tin and Copper Mining of Cornwall. The air in which the Miners work must necessarily be of a very impure character, as no matter how perfect the modes of ventilation may be carried out by the sinking of shafts, the conveying it to the different "winzes" or communicating shafts, and the "fast-ends" or *culs de sac*, in which the Miners work, is a most difficult matter. There is no doubt but that many more shafts might with advantage be sunk in some of the Mines, and especially is this the case in Cornwall, as it is by no means uncommon for only one to be in existence, consequently varying with the direction of the wind as to whether it is a "downcast" or "upcast" shaft. Again, under some circumstances, especially where the Mines extend much under the sea, the sinking of shafts is attended with great danger, but the beneficial effect derived from the same, when possible, is fully proved by the following paragraph extracted from the above-mentioned report.

"The ventilation of the Botallick Mine, which passes to a "distance of half-a-mile under the sea, was so imperfect, that "a diagonal shaft for conveying fresh air into the Mine, has been "sunk at an angle of $32\frac{1}{2}^{\circ}$ from the surface, to the extreme point "of the several principal levels. The Mine has by this means "been rendered much purer as regards its atmosphere, and like-"wise much cooler. Previous to the construction of this diagonal "shaft, candles would scarcely burn in some parts of the Mine, "and the temperature stood at about 87°."*

The heat in the Copper Mines was found to be very much more than that in Tin Mines, and "the temperature in the United

* Page 130, Third Report of Medical Officer of Privy Council.

"Consols Mine at Gwennap was said to be as high as 125°, and "the men can only work by short spells, and are constantly "supplied with cold water for drinking, which soon becoming hot "in the warm atmosphere of the Mine, is sent down from above "at very brief intervals. Steam was coming out of the shaft in "volumes at the time of inspection, and the temperature of the "hut at the top, in which evidence was obtained from some of the "older Miners, was sensibly affected by the warm air rising out of "the mine."*

There cannot possibly be two opinions as to the unhealthy influence such extreme heat must have on the Miners; and if, as suggested, any deleterious matter is absorbed into the system in the case of Lead and Copper Miners, this high temperature would most certainly facilitate such absorption.

Tin and Copper Mining is worked principally by blasting, and this again has a most injurious effect on the men. After each explosion, the air is filled with smoke and other products of combustion, and thereby rendered unfit for breathing. This state of affairs remains in existence in proportion to the ventilation of the Mine. In Iron and Coal Mines, where the ventilation is infinitely superior to that of the Tin, Lead, and Copper Mines, the sulphurous smell is soon carried away; whereas, in the latter, it will remain often for hours in existence, until, as the Miners appropriately express it, it "dies out."

The Miner's prospect in the Metalliferous Mines is certainly not very inviting. A temperature of 80° to 90°, not considered extraordinary, whilst it may even ascend to 125°. The heat so great that the Miner has often to work in a state of nature, and even then requires to be now and then revived by having cold water thrown over him; whilst the perspiration is so great, that men sometimes lose ten pounds in weight during a day's work. The atmosphere inhaled is but normal in ten per-cent of the Mines, whereas in over 65 per-cent it is noted as being exceedingly bad. And, finally, after the prostration of what may with some truth be called a hard day's work, the Miner has a height to climb, varying from 1,200 to 1,680 feet, as being the only mode of attaining the open air.

Lead Mining has now to be considered, and the circumstances under which the Miners are placed are a little different to those of Tin and Copper Mines. Lead Mines are generally worked by

* Same Report, page 131.

levels driven through the ground horizontally. As regards the ventilation, it is about on an equality with that of Tin and Copper Mines, but differs materially with the kind of strata through which the levels pass, as the following quotation from Dr. Greenhow's Report on Lead Mining testifies:—

"The Mines driven through limestone are, generally speaking, "the most wholesome; and their ventilation is frequently much "favoured by what are termed 'shakes,' or cavities in the strata, "which are sometimes of considerable size, but at others consist "of mere crevices or chinks. These shakes sometimes extend to "the surface; and though probably communicating by a very small "fissure with the open air, effect a communication between the "external atmosphere and that of the Mine. Mines driven in "limestone are also said to be less dusty than such as are driven "in either sandstone or shale.

"Mines driven through shale are the most unwholesome, and "the least pure as regards their atmosphere. They are much "dustier than the others, especially when the shale is very dry, "and are more liable to have their atmosphere vitiated by escapes "of gas. Though gas is said to be most frequently evolved from "shale, it is also sometimes met with in Mines driven through "other strata, particularly sandstone."*

The ore is obtained generally by blasting, though in the case of shale it is still sometimes got by the dusty process of picking. Again, before being carried out of the Mine, it has all to be broken into small pieces, which process has far from a healthy effect on those engaged in it, necessitating the respiration of air charged with more or less of the dust.

The diseases engendered by Lead Mining may be stated as Asthma and Chronic Bronchitis, though no doubt in those predisposed to consumption, it would very much tend to its development. All in any way connected with Mines, sooner or later suffer from Asthma, though this is found to take place at an earlier age among Lead and Copper Miners.

As regards Mining generally, it will be perceived that the unhealthy influences are first, the impure air, which the five classes of Iron, Coal, Tin, Lead, and Copper Miners are obliged to inhale, and which influence, on account of the superior artificial modes of ventilation adopted in the Mines of the two first classes, renders them much more healthy than the remainder. Secondly, the sulphurous smoke and gases produced by blasting, to which all five

* Page 139, Third Report Medical Officer of Privy Council.

classes it is true are liable, but which again, as regards Iron and Coal Mines, is generally very soon removed by the superior ventilation; whereas in Lead and Copper Mines, it may remain for hours in existence. In fact, in some Lead Mines, it is the custom of the Miners for this reason to fire as many blasts as possible just before dinner, so as to leave the atmosphere pretty clear for working on their return. And lastly, the extreme heat to which some classes of Miners are exposed, such heat being greatest in the case of Copper Mines.

During Dr. Greenhow's investigations, it was found to be the general impression of both Managers and Miners, that those men who lived at a distance from the Mine were more healthy than those located in its immediate neighbourhood. This very clearly shows the healthy influence of the exercise that these Miners were necessitated to take each day before and after their day's work, which undoubtedly counteracts, to some extent, the bad effects engendered by their occupation during working hours.

A similar opinion as to the better health of those who lived at a distance from their employment prevails to a great degree in many of the factory occupations; and would suggest, therefore, the feasibility of providing dwellings for the operatives, where possible, at a little distance from their work, thereby occasioning a certain amount of daily exercise, which could not but prove most beneficial to them, and ameliorate, in some degree, the noxious effects of their occupation.

The mortality of those working or dealing in Iron, Tin, Lead, and Copper.

It has been stated before how comparatively healthy Iron Miners are to Tin, Lead, and Copper Miners; and the object of the next combination is to show that nearly all engaged in trades or occupations in connection with the three latter metals, experience a higher rate of mortality than those working or dealing in Iron. (*See Table B.*)

Observing the pursuits in connection with Iron, it will not fail to be perceived that not in one single instance is the mortality for the period of life, 25 to 65, above that for England and Wales generally. Ironmongers experience, during this period, the lowest mortality, and the two manufactures of Iron and Nails come next, then appears Iron Miners, and last of all Blacksmiths. This is a most striking result, for though all the occupations, it is true, are in connec-

tion with Iron, yet, nevertheless, the employments are in their nature widely different. Whilst the Iron Miner is subject to the inhalation of an atmosphere always somewhat foul, and liable moreover to many accidents, the Ironmonger is certainly exposed to neither of these noxious influences. Again, whereas Blacksmiths are exposed to varied temperatures, and their trade involves great muscular exertion, the Ironmonger's necessitates very little. At the older ages, the effects of the more unhealthy occupations begin to develope themselves, and Iron Miners and those engaged in the Iron Manufacture, experience the highest mortality, whilst Ironmongers have the least.

Tin Miners, it was shown, enjoyed a middle position between Iron and the more unhealthy Miners; and in the above Table B it will be noticed that the mortality among those working or dealing in Tin is likewise higher than for those in connection with Iron. The only exception is the Tin Manufacture, but the large body of men employed therein, under the designation of Tin-platers, who are a very healthy class, accounts for this. Those engaged in the Tin Manufacture appear as the most healthy, then Tinkers, and lastly Tin Miners, and Whitesmiths. In the analysis of the Friendly Society results also, it may be mentioned, Blacksmiths were found to be materially more healthy than Whitesmiths. In this class also, in advanced life, the Miners display the greatest mortality.

In the case of Copper the differences of mortality are more varied, in consequence of the high mortality of Copper Miners. Coppersmiths appear as the most healthy, the Copper Miners as the least; in fact, at each period of life the Miners exhibit the highest mortality. At the older ages the mortality in each occupation is over three per-cent, being nearly five per-cent amongst the Copper Miners. These Miners, however, have already been shown to be the least healthy of all Miners; the extreme heat they are exposed to having a most deleterious effect.

In the last class, namely, that of Workers in Lead, Plumbers Painters and Glaziers present the least mortality, Lead Miners the highest. Painters are much subject to colic, and poisoning by absorption of lead; but the health of this trade has undoubtedly been much improved by the wearing of working clothes of a washing material; so that they can frequently be cleansed, and moreover taken off after the day's work is finished. Formerly, a suit was worn until it was so coated with paint as to be no longer wearable; and what is worse, kept on, it is to be feared, even after working

hours, thus tending to impregnate everything. That lead does exert an injurious influence, is well illustrated by the effect that the type has on Compositors. An elaborate inquiry into the health of this class will be found in the Sixth Report of the Medical Officer of the Privy Council. As among the occupations in connection with Copper, so with Lead, the mortality in advanced life is, in every instance, more than three per-cent.

To summarize the results, it would appear that the difference of the mortality of occupations connected with various metals is very considerable, and would tend to the conclusion that the metal in which persons deal or work has a most important effect on health.

For instance, it appears, regarding the period of life 25 to 65, that the mortality for each of the four metals, no matter how different in other respects the occupations connected with each metal may be, is confined within, speaking generally, certain bounds, and these are as under:—

Metal.	Mortality.
Iron.	12·6 to 13·8 per 1,000
Tin.	15·6 „ 16·8 „ „
Lead.	18·3 „ 20·6 „ „
Copper.	17·1 „ 24·7 „ „

In the first three metals, the mortality is restricted within remarkably small bounds, but as regards Copper the great variation arises on account of the very high mortality of the Copper Miners, otherwise it would be confined between 17·1 and 18·5 per 1,000.

It may be concluded therefore that Iron is the most healthy metal to work or deal in, Tin the next, and, if Copper Miners be excluded, Copper comes next, otherwise Lead.

The influence that various Manufactures have upon Health.

The next Table that will be presented is compiled so as to display the influence of various manufactures upon the health of those engaged in them, and the manufactures in the four metals, it will be observed, follow in precisely the order laid down in the last combination, namely, the Iron Manufacture as being the most healthy, the Tin next, and then the Copper and Lead, the two latter being much alike. *(See Table C.)*

The Copper, Lead, and Earthenware Manufactures appear as the most unhealthy, these three alone exhibiting a mortality in excess of three per-cent for the older ages. At the younger ages (25 to 45) these three in conjunction with the Glass Manufacture are the only

ones out of the nine different manufactures under review, which present a mortality in excess of that for the population at large. Taken altogether, the Paper and Nail Manufactures are the most healthy, and then follow the Tin, Iron, Brass, Glass, Copper, Lead, and Earthenware Manufactures.

These different manufactures contain, however, branches which are in some cases very injurious. In the Brass Manufacture these occupations are principally the Brass Founders and Casters. As regards Brass Founders, they are much exposed to the inhalation of very fine dust, given off from the different moulds, especially charcoal dust. Again, in the process of casting, when the Copper is melted, very obnoxious fumes are generated, and these are greatly increased on the addition of the Zinc, as the atmosphere becomes then impregnated with Oxide of Zinc, which has a very deleterious effect on the men's health. Hardly any fumes have such a prejudicial effect upon animal and vegetable life as those given off in this process. The country sometimes for miles round is affected by them. The result in so far as relates to those engaged in these avocations, is to render them very subject to Asthma and Bronchitis.

Iron Founders and Casters are also unhealthy occupations, but not nearly to such an extent as the same classes in the Brass Manufacture; as in the process of Iron Moulding such fine materials are not used, and therefore the danger of inhaling the dust is not so great. Moreover, in Iron casting the fumes generated are not by any means so obnoxious. There are some other branches of the Iron Manufacture which deserve notice on account of the workmen being liable to abnormal influences, such as Puddlers, who are exposed to extreme heat with very exhausting labour; also Enamellers, who are placed under much the same circumstances as many engaged in the Potteries, in so far as regards their liability to breath an air more or less loaded with glass; and lastly, Turners, whose work of necessity creates a great deal of iron dust, but which being generally of too heavy a description to float long in the atmosphere, soon falls to the ground, and thus in a very great measure precludes the air becoming impregnated with it.

The Nail Manufacture is carried on principally at the homes of the operatives, and possesses nothing essentially injurious to health. The Lead Manufacture, on the other hand, includes one very unhealthy branch, namely Typefounders.

The Earthenware Manufacture, the most unhealthy of all in the table, has now to be considered. From a careful analysis of

the mortality of members of Friendly Societies following the avocation of a Potter, it was found that the mortality corroborated the results now submitted. Whatever the circumstances rendering a Potter's life so unhealthy may be, it will not fail to be noticed that the injurious influence is most prominently shown at the ages 45 to 65, whereat the mortality is not far short of four per-cent.

Amongst those engaged in the Potteries there are some branches which are more particularly dangerous, as exerting an injurious influence upon health; such as Slip-makers, Mould-makers, Potters (properly so called), Turners, China Scourers, and Decorators. In all these various branches, pulmonary diseases, induced by the great quantities of dust the workers inhale, are found to be very prevalent; and each branch, according to Dr. Greenhow, is more or less unhealthy, in proportion as the workmen are exposed to one or more of the following influences, namely, first, to the inhaling an air charged, more or less, with fine irritating dust; secondly, to exposure to a dry hot atmosphere or to a hot moist atmosphere, or great changes of temperature; and lastly, to habitually assuming a constrained attitude when working.

The stoves for drying the earthenware are placed in the workshop, thus causing a very high temperature, and "the ware is "carried into the stoves by boys, who are very young, and are yet "kept running to and fro all day, thereby filling the atmosphere "of the shops with dust. The quantity of dust varies according "to the cleanliness of the place. Some workshops are swept daily, "others only once a week, and of course the operatives employed "in the latter are more exposed to inhale dust than those in the "former. The temperature of the workshops depends partly upon "the sufficiency of the supply of moulds. When the men are well "supplied with them, it is not necessary to hasten the process of "drying, and the stoves need not be so highly heated. When, on "the other hand, there is a deficiency of moulds, the Potters "endeavour, by way of compensation, to hasten the process of "drying, in order that the moulds may again be soon ready for "use."*

The most unhealthy class in the Potteries are undoubtedly China-scourers, a branch followed always by women, who remove the roughness of the china, after it has been baked, with sandpaper, thereby creating great quantities of fine flint dust. The better the

* Page 107, Third Report of Medical Officer of Privy Council.

china, the more injurioús the effect it has, as the dust is finer. Concerning this branch of the Earthenware Manufacture Dr. Greenhow states that in one establishment visited, only one China-scourer had worked so long as three years at the occupation, and that "in a third Pottery, a woman who had worked ten years at " the occupation, asserted that about twelve other scourers in the " same shop had died since she entered it."

It is found that China-scourers very soon become Asthmatical after working for a short period, no matter how healthy before.

These quotations very plainly show the injurious influences to which Potters are subject; and in all the processes, the operatives are exposed to the inhaling the fine dust with which the air of the different workshops is charged, and which dust, the finer it is, the longer it floats in the atmosphere, and the more dangerous it becomes. If the work-rooms were constantly swept, the health of the operatives could not but be much improved, as the moist dough of the Potters, which is continually dropping on the floor, is very soon, on account of the high temperature of the rooms, dried; and then, easily becoming converted into powder, is perpetually being raised into the atmosphere by the frequent locomotion in the room. It would be also a decided improvement if the stoves could be removed further from the operatives, as they render the temperature of the workshops very high.

Some of the evils incidental to the Earthenware Manufacture have, it is said, been aggravated by the circumstance that in the present day there is never any cessation of work during the winter. Formerly, it was the custom to annually close the Potteries for some weeks in frosty weather, and this respite from labour was found favourable, in some degree, to the Potters recovering from the noxious effects engendered by their occupation. Improvements have lately, however, been introduced, by which the Potteries are enabled to continue in operation all the year round, and thus any advantage that may formerly have arisen from the winter's respite is now lost to the operatives.

It is much to be regretted that the Potters aggravate the noxious effects of their occupations by being much addicted to drinking; as in another part of this paper it will be shown that nothing exercises such an injurious influence upon health. It is, however, some consolation to find from Dr. Greenhow's Report, that an improvement is thought to be taking place in this respect, and that the Potters are less given now than formerly to irregular habits.

The principal disease engendered by the Earthenware Manufacturers may be stated as Chronic Bronchitis; though, no doubt in those predisposed to Phthisis, it would tend to its developement. As for "Potter's" or "Miller's" Asthma, as it is called, Dr. Greenhow believes it to be essentially a form of bronchial irritation.

The mortality among occupations in connection with Railways.

The next table presents a very different class of results, displaying the mortality amongst those engaged in connection with Railways. The sum total of all the occupations embraced in the accompanying table is nearly 68,000 men between the ages of 25 and 65; so that, altogether, those engaged upon Railways form a numerous body. *(See Table D.)*

The Company's Officers, Clerks, and Station Masters, it will be observed, experienced the lowest rate of mortality, and seeing that there is nothing much in their employment exposing them to danger, this might be expected. Labourers, Platelayers, and Navvies, come next on the list; but in their case, being subject to numerous accidents arising from the nature of their employment, their mortality, though less than that of the whole population, is nevertheless in excess of that experienced by Labourers in general. Servants, Porters, and Gate-keepers, being even more exposed to accidents than the preceding class, and the occupation moreover of itself not being so healthy, have a mortality slightly higher than that for Labourers, Platelayers, and Navvies. Engine Drivers and Stokers are certainly the class in which the danger arising from accidental causes is at its highest intensity, and, as therefore might be expected, the death-rate is greater than in any of the former combinations, being for the 40 years of age, 25 to 65, at the rate of 16·3 per 1,000.

Thus the mortality of those engaged on Railways is, notwithstanding the peculiar risk to which many of the occupations are exposed, less than would probably have been expected, as, if the sum total of all is taken, the mortality, it will be perceived, is but slightly in excess of that for the kingdom generally.

The most striking result in the table is the rate of mortality at the earlier ages in all those combinations which are more particularly liable to risk from accident. This mortality, it will be noticed, increases from 11·9 per 1,000 amongst Labourers, Platelayers, and Navvies, to 12·7 among Servants, Porters, and Gatekeepers, and finally reaches its maximum of 14·7 per 1,000 in the case of Engine Drivers and Stokers.

This would almost imply therefore, as the mortality at the older ages assimilates more to that of the country generally, that the only injurious influence of those engaged on Railways is to be attributed to the risk arising from accidents.

The Mortality of Occupations in connection with Animal Food.

Butchers have always been regarded as a somewhat unhealthy class, though as yet no reason for their being so has been assigned. The results, however, of this investigation fully bear out the unhealthfulness of their calling, and what is more peculiar, of that of all trades in connection with animal food.

The following table exhibits the facts in connection with this subject. (*See Table E.*)

The deaths in the three out of the four trades were, for the 40 years of age, 25 to 65, at about the rate of 17 per 1,000 living; Poulterers, the remaining trade, being subject to a mortality as high as 21 per 1,000. Provision Curers are, on the whole, the most healthy, Poulterers the least so, Butchers and Fishmongers closely agreeing.

As relates to Butchers, having the experience of several thousand years of life of members of this occupation who had been members of Friendly Societies, I have had the same carefully analysed, and the results corroborate the above in a remarkable manner; the rate of mortality for the 40 years 25 to 65 appearing as 17·9 per 1,000 living.

As to the cause of this high mortality amongst all trades in connection with animal food, several might, with some degree of probability be advanced. First of all, the inhalation of an atmosphere always to some degree impregnated with animal matter, must have a very deleterious effect. Secondly: the latter cause in conjunction with the manner in which they are exposed to the weather, and the lack of exercise. And, lastly, it might in a great degree be influenced by the partaking of an undue amount of animal food. Moreover, there would of necessity occasionally be much putrescent animal matter, and therefore they would be more than ordinarily exposed to fevers.

The influence of Domestic Service.

As to whether there is any material difference in the mortality of those engaged in Domestic Service in a given occupation, and those that are not so situated, but are nevertheless of much the same occupation, is the subject of the next table. The circumstances under which the two different classes are placed are, it is

true, in many respects very dissimilar; for it must be concluded that those engaged in Domestic Service will generally be both better fed, housed, and clothed, than those who are not so circumstanced. It remains to be proved however whether this of itself is sufficient to account for the immense difference appearing between the two results. (*See Table F.*)

Gardeners, Grooms, Servants, and Coachmen are in every case, and throughout all periods of age, much more healthy in Domestic Service than those who not so placed, the main distinction amongst the four occupations being the ratio of such difference. In the case of Gardeners there is not a very great variation between the mortality of those in Domestic Service, and those not so situated; but when the other three employments are regarded the results are most striking.

Let us first observe these three occupations, as illustrated by those employed in Domestic Service. Grooms appear much more healthy than Coachmen, but this may arise perhaps from the duties of the former necessitating more vigorous exercise, whilst at the same time they are not exposed to the inclemencies of the weather to such a degree as Coachmen. It is the lower mortality of Grooms at the older age that constitutes the main difference between the two, so that apparently it is at these ages that the dissimilarity of employment most forcibly displays itself. Male Domestic Servants appear as experiencing a mortality somewhat less than that of Coachmen.

If now the same three occupations, namely Grooms, Servants, and Coachmen, as exemplified by those *not* in Domestic Service, be referred to, a very high mortality is at once seen to be experienced, and moreover, instead of the great difference in the death-rate as amongst the same employments in Domestic Service a uniform one is maintained, being about 22 per 1,000 for the period of life, 25 to 65. At the early ages, although in every instance the mortality is extremely high, the relative position of the occupations in respect of their respective mortalities is not quite obliterated, but when the more advanced ages are regarded, all similarity as to the positions of the same avocations in Domestic Service immediately disappears, for though, as has already been exhibited in Domestic Service, Coachmen experience a higher mortality than Servants, and one much in excess of that for Grooms; yet, with these same occupations out of Domestic Service, all exhibit a mortality of the extremely high rate of 36 deaths per 1,000 living.

There is, therefore, some predominating influence which accounts for this great difference in the health of these employments when not in Domestic Service; and some influence, moreover, which is sufficiently powerful as to set aside the effect which occupation itself would appear to exercise in the case of Grooms. As regards those not in Domestic Service, Grooms include Horsekeepers, and Jockeys, and Coachmen, Cabmen, and Flymen; whereas, Servants, non-domestic, are composed solely of Male Inn-Servants. Granted, that circumstances under which the occupations in Domestic Service are placed, are entirely different to those of the employments not so situated, and that the latter are neither so well fed, housed, or clothed; yet can this of itself possibly account for the two very different rates of mortality? If so, why is it that Gardeners not in Domestic Service are not likewise affected?

Is it not rather that the classes of Grooms, Servants and Coachmen, not in Domestic Service, are much addicted to irregular and drinking habits, as the very high mortality at the younger ages would tend to prove? For how can this mortality be possibly accounted for by anything peculiar to the several occupations, since, as will be presently shown, in no other trades or occupations, *except* those in connection with drinks and stimulants, does such a rate prevail?

Confirming to some degree this supposition is the fact, that of the three occupations under consideration, that one which exhibits the maximum mortality at these ages, is that employment which is more particularly brought in connection with Drinks and Stimulants, namely, Male Inn-servants. For whereas in the other two classes, namely, Grooms and Coachmen, the mortality was as high as at the rate of 15·5 per 1,000; amongst Inn-servants it was 17·5 per 1,000, or, peculiarly enough, the identical rate prevalent at these ages amongst Publicans, and Licensed Victuallers, and Inn and Hotel Keepers.

The influence of Drinks and Stimulants upon Health.

Though no advocate of teetotalism, the next table certainly presents facts of the utmost importance as to the influence of drinks and stimulants upon health. (*See Table G.*)

In every one of the different classes the mortality, it will be observed, is very high, being lowest for Beersellers. Wine and Spirit Merchants, though showing a much higher mortality than

Beersellers, are next on the list; and with hardly any perceptible difference come Publicans and Licensed Victuallers. Inn-keepers and Hotel-keepers appear as the least healthy, the mortality in their case being, for the period of life 25 to 65, at the extremely high rate of 26·8 per 1,000 living.

In no other class of results is such a high death-rate presented as amongst the various occupations connected with drinks and stimulants; for combining the different classes together, the mortality, it will be observed, for the 40 years intervening between 25 and 65 would be at the rate of over 24 deaths per 1,000, whereas the mortality for the whole population of England and Wales during the same period of life was at the rate of 15 per 1,000 living, and amongst Gardeners in Domestic Service under 8 per 1,000. In each of the occupations connected with drinks and stimulants, is the mortality at the older ages over 3 per-cent, being highest amongst Inn and Hotel-keepers, and lowest amongst Beersellers.

Some interesting observations as to the difference displayed between the mortality of those connected with the different drinks and stimulants may be exhibited.

Mr. Neison some years ago made a very elaborate enquiry into the mortality prevailing among intemperate lives. The investigation extended over 6,111 years of life, and none were included, it may be mentioned, but those who were decidedly addicted to drinking habits. The results were compiled, and presented in a concise form in his valuable "Contributions to Vital Statistics," and the following, as relating to the question in point, is extracted therefrom.

"The facts collected in the preceding Schedule required an "enumeration of the peculiar features of the intemperance in "respect to the favourite beverage, and it is curious to remark the "influence of the different kinds of drink on the duration of life.

"The average duration of life after the commencement of the "intemperate habits, is—

"Among Beer drinkers	21·7 years.
" " Spirit "	16·7 "
"And among those who drink both Spirits " and Beer indiscriminately .	16·1 "

"and consequently the rate of mortality will be—

"Among Beer drinkers . . .	4·597 per cent.
" " Spirit "	5·996 " "
" " Mixed "	6·194 " "

"Intemperate indulgence in the use of distilled liquors is hence "more hurtful to health than the like use of fermented liquors

" but the immoderate use of both combined is more injurious than " the exclusive use of the one kind only."

These observations are most remarkably confirmed in the present table where Beersellers, it will be perceived, are decidedly much more healthy than either Spirit Merchants, or Publicans and Licensed Victuallers. Again, the little difference appearing in Mr. Neison's results between the duration of life after the commencement of intemperate habits amongst those who confined themselves to spirits only, and those who drink both spirits and beer indiscriminately, is likewise exhibited in the slight difference between the mortality of Wine and Spirit Merchants, and Publicans and Licensed Victuallers.

Another most peculiar feature is, that omitting Beersellers, each of the other occupations in the table shows at the younger ages a mortality exceeding 17 per 1,000. This is, it is needless to observe, an exceedingly high mortality, and for that period of life in no other trade or occupation compiled from the experience of England and Wales for the years 1860, 1861, is such a high rate shown, with the exception of the two occupations of Male Inn Servants, and Brewers, both which avocations it may almost be said are placed under much the same circumstances in respect of the ruling influence at work in causing their unhealthfulness, as the trades contained in the present table. It would appear, therefore, that in those occupations connected with drinks and stimulants only, does the mortality at the younger ages attain as high a rate as 17 per 1,000 living; for notwithstanding all the unhealthy and injurious influences to which those engaged in Mining, the Earthenware, and several of the obnoxious Factory occupations have been shown to be exposed, in none of these pursuits is the mortality for the same period of life above 14 per 1,000 living.

The Mortality among the Professions.

The relative health of the different Professions is the subject of the next combination. (*See Table H.*)

The Clergy, as is well known, have long been supposed to be an exceedingly healthy body, and the accuracy of this supposition the table displays. The Legal Profession follows next in order, the mortality in this being on a par with that for the country generally. The Medical Profession, on the other hand, is not, as might be expected, so healthy as either of the preceding, its mortality for the period of life, 25 to 65, being at the rate of over 17 per 1,000.

This increased mortality amongst the members of the Medical Profession will be found accounted for by the high rate experienced at the younger ages, whereat their avocation exposes them to much more risk than is the case in the Clerical or Legal Professions. The Clergy though on the whole very healthy, are, it will not fail to be noticed, more particularly so at the older ages.

Dr. Caspar of Berlin in his enquiry as to the longevity of the different Professions, corroborates the above results as to their respective order in a point of health, placing Clergymen first, then Lawyers, and lastly Medical Men.

When divided into its component parts, the Clerical Profession's most healthy members will be noticed to be Protestant Ministers,* then Clergymen of the Church of England, and lastly Roman Catholic Priests. This latter class experienced a much higher mortality than either of the other two; but this may perhaps be attributed to the very different status of the members of their flock, and the risk consequently they are exposed to in visiting and attending them.

In the Legal Profession the higher branch, consisting of Barristers, Advocates, Special Pleaders, and Conveyancers, are subject to a rate of mortality only much the same as that experienced by Clergymen and Ministers, but their lower brethen of Solicitors and Attornies are by no means so favoured, having a death-rate in excess of that of the population at large.

In the Medical Profession also, this result occurs of the higher branch experiencing a much less rate of mortality than the lower, Physicians being subject to a mortality of only 12·6 per 1,000 for the 40 years intervening between 25 and 65, whereas Surgeons and Apothecaries have for the same period of life a mortality of 18·6.

This feature of the higher branch in the two Professions of Law and Physic being much more healthy than the remainder, is worthy of note, as it will be observed that in both instances the principal difference occurs at the younger ages, where, in both Professions, the mortality in the one branch is about double that of the other.

My paper having already extended to such length will not admit of the consideration of any further results, though interesting facts in connection with many other pursuits were obtained.

* Dissenting Ministers of every persuasion but the Roman Catholic are included under this designation.

There are, however, one or two points to which, before concluding, I would wish to draw attention.

First then as to some peculiar combinations bearing on the influence of occupation that have appeared in the works of authors of high standing.

Mr. Neison in his "Contributions to Vital Statistics" grouped together numerous trades according to the circumstance of their being out-door or in-door occupations, and requiring little or great exercise. From this classification it appeared that out-door occupations requiring little exercise were most unhealthy, then in-door trades with little exercise, the most healthy combination being out-door occupations with great exercise. In other words that those avocations demanding great exercise, whether located indoors or out, were superior in health to such pursuits as necessitated little exercise.

The other arrangement was that adopted by Mr. Finlaison in his Report on the Friendly Society Returns. In this classification the occupations were arranged as follows:—Light labour with exposure to the weather, light labour without exposure to the weather, and heavy labour under each of the above conditions. Heavy labour with exposure to the weather appeared to be the healthiest class, and next to that, light labour with exposure to the weather, thus showing that, under both conditions of labour, were those with exposure to the weather most healthy.

To test the accuracy of the results obtained by grouping trades or occupations together as above, it was found on examining the mortalities of such trades as each of the above gentlemen stated formed the basis of the different combinations, that in each section occupations were included whose mortalities were widely divergent; in fact, in every respective section were representatives found, speaking generally, of the most healthy and unhealthy avocations.

It will, doubtless, have been noticed that in nearly every occupation in which any particular disease has been mentioned as being engendered by such pursuit, that disease has been generally one which would come under the classification of Pulmonary diseases. The diseases embraced under this heading are as follows, namely: Consumption, and diseases of the respiratory organs, the two principal of which are Bronchitis and Pneumonia, the remainder being Asthma, Pleurisy, Laryngitis, and "Diseases of the Lungs."

The extent to which these diseases are prevalent is not generally known; but it will suffice to show the importance of the question, to mention, that, of the 4,455,436 deaths from all causes in England

and Wales, during the 10 years, 1856-65, no less than 1,160,904, or over 26 per-cent, were from Pulmonary diseases alone; thus, more than a quarter of those dying every year subsiding to these diseases. Consumption was most fatal, and then Bronchitis and Pneumonia; but of recent years, whether from an improved diagnosis of disease amongst the Medical faculty, or from some other cause, the percentage of deaths returned for Consumption and Pneumonia has been rapidly decreasing; whilst an immense increase has taken place in those from Bronchitis.

As I have demonstrated elsewhere*, the differences in the mortality of the various analyses of the returns of Friendly Societies that have been made, is to be attributed solely to the influence of occupation. For, inasmuch as the average mortality of the experience as a whole is founded on a combination of occupations, it is very clear that it must entirely depend upon the percentage of healthy or unhealthy trades in such combination, as to whether the mortality shall be low, or otherwise. In fact, so far as concerns mortality, instead of the different returns more or less contradicting one another, as is generally thought to have been the case, when carefully analysed and regarded in their component parts, they will be found to corroborate one another remarkably.

In conclusion, after the facts adduced in the present paper, it may safely be asserted, I think, that occupation exerts a most material influence on the health of our population, and one calculated to throw much light on many obscure and vexed questions connected with the sanitary condition of our large towns. Few questions could better occupy the attention of our legislators than the consideration of in what respects many of the occupations of the working classes could be rendered more salubrious; and now that measures of a home description are essentially the order of the day, it is to be hoped Parliament will soon devote itself to the adjustment of this subject.

* "Present knowledge of the Mortality and Sickness of Members of Friendly Societies." Submitted for the consideration of the Royal Commission on Friendly Societies. *Thomas Brettell & Co.* 1871.

TABLE A.

Table showing the Mortality among those engaged in Underground Occupations.

Occupations.	Males—Ages.									
	25 to 65.			25 to 45.			45 to 65.			25 to 65.
	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	*Mean Mortality.*
Iron Miners	12,346	338	1·37	9,437	185	·98	2,909	153	2·63	1·80
Coal Miners	128,741	3,811	1·48	95,600	2,095	1·10	33,141	1,716	2·54	1·82
Miners in Iron, Coal, Tin, Lead, and Copper	167,432	5,241	1·56	123,295	2,734	1·11	44,137	2,507	2·84	1·97
Tin Miners	6,567	211	1·61	4,404	75	·85	2,163	136	3·14	1·99
Miners in Iron, Tin, Lead, and Copper	38,691	1,430	1·85	27,695	639	1·15	10,996	791	3·60	2·37
Lead Miners	11,031	449	2·03	7,731	207	1·34	3,300	242	3·67	2·50
Copper Miners	8,747	432	2·47	6,123	172	1·40	2,624	260	4·95	3·17

TABLE B.

Table showing the Mortality among those working or dealing in Iron, Tin, Lead, and Copper.

Occupations.	Males—Ages. 25 to 65. Living in 1861.	25 to 65. Deaths in 1860 and 1861.	25 to 65. *Annual Mortality per-cent*	25 to 45. Living in 1861.	25 to 45. Deaths in 1860 and 1861.	25 to 45. *Annual Mortality per-cent.*	45 to 65. Living in 1861.	45 to 65. Deaths in 1860 and 1861.	45 to 65. *Annual Mortality per-cent.*	25 to 65. *Mean Mortality.*
IRON—										
Ironmongers	5,371	135	1·26	3,639	74	1·02	1,732	61	1·76	1·39
Iron Manufacture	68,593	1,749	1·27	52,773	912	·86	15,820	837	2·64	1·75
Nail Manufacture	9,001	238	1·32	5,337	82	·77	3,664	156	2·13	1·45
Iron Miners	12,346	338	1·37	9,437	185	·98	2,909	153	2·63	1·80
Blacksmiths	63,223	1,751	1·38	42,330	801	·95	20,893	950	2·27	1·61
TIN—										
Tin Manufacture	5,557	146	1·31	3,945	71	·90	1,612	75	2·33	1·61
Tinkers	4,291	134	1·56	2,887	66	1·14	1,404	68	2·42	1·78
Tin Miners	6,567	211	1·61	4,404	75	·85	2,163	136	3·14	1·99
Whitesmiths	5,831	196	1·68	3,874	81	1·04	1,957	115	2·94	1·99
COPPER—										
Coppersmiths	1,108	38	1·71	757	15	·99	351	23	3·28	2·13
Copper Manufacture	2,377	88	1·85	1,615	42	1·30	762	46	3·02	2·16
Copper Miners	8,747	432	2·47	6,123	172	1·40	2,624	260	4·95	3·17
LEAD—										
Plumbers, Painters, and Glaziers	48,681	1,786	1·83	34,492	818	1·19	14,189	968	3·41	2·30
Lead Manufacture	2,146	83	1·93	1,465	41	1·40	681	42	3·08	2·24
Lead Miners	11,031	449	2·03	7,731	207	1·34	3,300	242	3·67	2·50

TABLE C.

Table showing the Mortality among those engaged in various Manufactures.

Occupations.	Males—Ages.									
	25 to 65.			25 to 45.			45 to 65.			25 to 65.
	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	*Mean Mortality.*
The Iron Manufacture	68,593	1,749	1·27	52,773	912	·86	15,820	837	2·64	1·75
" Paper "	4,232	110	1·30	2,708	49	·90	1,524	61	2·00	1·45
" Tin "	5,557	146	1·31	3,945	71	·90	1,612	75	2·33	1·61
" Nail "	9,001	238	1·32	5,337	82	·77	3,664	156	2·13	1·45
" Brass "	7,924	218	1·38	5,843	113	·97	2,081	105	2·52	1·74
" Glass "	7,444	236	1·58	5,329	135	1·27	2,115	101	2·39	1·83
" Copper "	3,485	126	1·85	2,372	57	1·30	1,113	69	3·02	2·16
" Lead "	2,146	83	1·93	1,465	41	1·40	681	42	3·08	2·24
" Earthenware Manufacture	13,423	528	1·97	9,710	237	1·22	3,713	291	3·92	2·57

TABLE D.

Table showing the Mortality among those engaged in Occupations in connection with Railways.

Occupations.	Males—Ages.									
	25 to 65			25 to 45.			45 to 65.			25 to 65.
	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	*Mean Mortality.*
Railway Officers, Clerks, Station Masters	8,250	211	1·28	6,543	132	1·01	1,707	79	2·31	1·66
" Labourers, Platelayers and Navvies	32,048	913	1·42	24,531	584	1·19	7,517	329	2·19	1·69
" Officers, Clerks, Station Masters, Porters, Gatekeepers, Police, Labourers, Platelayers, Navvies Engine Drivers and Stokers . .	67,842	1,989	1·47	53,742	1,327	1·23	14,100	662	2·35	1·79
" Officers, Clerks, Station Masters, Servants, Porters, Gatekeepers, Police, Engine Drivers and Stokers	35,794	1,076	1·50	29,211	743	1·27	6,583	333	2·53	1·90
" Servants, Porters and Gatekeepers	19,248	584	1·52	15,555	396	1·27	3,693	188	2·54	1·90
" Engine Drivers and Stokers	7,129	232	1·63	6,188	182	1·47	941	50	2·66	2·06

TABLE E.

Table showing the Mortality among those engaged in Occupations in connection with Animal Food.

Occupations.	Males—Ages. 25 to 65. Living in 1861.	25 to 65. Deaths in 1860 and 1861.	25 to 65. *Annual Mortality per-cent.*	25 to 45. Living in 1861.	25 to 45. Deaths in 1860 and 1861.	25 to 45. *Annual Mortality per-cent.*	45 to 65. Living in 1861.	45 to 65. Deaths in 1860 and 1861.	45 to 65. *Annual Mortality per-cent.*	25 to 65. *Mean Mortality.*
Provision Curers	5,989	201	1·68	3,448	66	·96	2,541	135	2·66	1·81
Fishmongers	6,696	233	1·74	4,292	116	1·35	2,404	117	2·43	1·89
Butchers	40,393	1,405	1·74	26,413	633	1·20	13,980	772	2·76	1·98
Poulterers	1,517	64	2·11	912	28	1·53	605	36	2·97	2·25

TABLE F.

Table showing the Mortality among those engaged in Domestic Service.

Occupations.	Males—Ages.									
	25 to 65.			25 to 45.			45 to 65.			25 to 65.
	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	*Mean Mortality.*
DOMESTIC—										
Gardeners	10,555	167	·79	6,728	56	·42	3,827	111	1·45	·93
Grooms	7,717	152	·93	6,186	99	·80	1,531	53	1·73	1·26
Servants	28,876	787	1·36	20,933	413	·99	7,943	374	2·35	1·67
Coachmen	10,287	302	1·47	7,410	146	·98	2,877	156	2·71	1·84
NON-DOMESTIC—										
Gardeners	53,227	1,160	1·09	28,414	348	·61	24,813	812	1·64	1·12
Grooms	23,488	1,024	2·18	16,108	488	1·51	7,380	536	3.63	2·57
Servants	10,909	472	2·16	8,468	297	1·75	2,441	175	3·58	2·66
Coachmen	21,863	974	2·23	15,112	481	1·59	6,751	493	3·65	2·62

TABLE G.

Table showing the Mortality among those engaged in various Occupations connected with Drinks and Stimulants.

Occupations	Males—Ages. 25 to 65.			25 to 45.			45 to 65.			25 to 65.
	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	*Mean Mortality.*
Beersellers	10,428	429	2·06	5,752	146	1·27	4,676	283	3·03	2·15
Wine and Spirit Merchants..............	5,483	256	2·33	3,440	126	1·83	2,043	130	3·18	2·50
Publicans aud Licensed Victuallers...	27,671	1,321	2·39	15,157	523	1·72	12,514	798	3·19	2·45
Beersellers, Wine and Spirit Merchants, Publicans and Licensed Victuallers, and Inn and Hotel Keepers	62,316	3,010	2·41	33,945	1,138	1·68	28,371	1,872	330	2·49
Publicans, Licensed Victuallers, Inn and Hotel Keepers	46,405	2,325	2·50	24,753	866	1·75	21,652	1,459	3·37	2·56
Inn and Hotel Keepers	18,734	1,004	2·68	9,596	343	1·79	9,138	661	3·62	2·70

TABLE H.

Table showing the Mortality among those engaged in the different Professions.

Occupations.	Males—Ages.									
	25 to 65.			25 to 45.			45 to 65.			25 to 65.
	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	Living in 1861.	Deaths in 1860 and 1861.	*Annual Mortality per-cent.*	*Mean Mortality.*
In the Clerical Profession	24,418	509	1·04	13,786	144	·52	10,632	365	1·72	1·12
" Legal "	12,749	384	1·51	7,050	129	·91	5,699	255	2·24	1·57
" Medical "	12,252	433	1·77	6,649	170	1·28	5,603	263	2·35	1·81
THE CLERICAL PROFESSION—										
Protestant Ministers	6,539	132	1·01	3,605	39	·54	2,934	93	1·58	1·06
Clergymen	16,766	342	1·02	9,458	91	·48	7,308	251	1·72	1·10
Roman Catholic Priests	1,113	35	1·57	723	14	·97	390	21	2·69	1·83
THE LEGAL PROFESSION—										
Barristers	2,802	61	1·09	1,645	19	·58	1,157	42	1·81	1·19
Solicitors, Attorneys	9,947	323	1·62	5,405	110	1·02	4,542	213	2·34	1·68
THE MEDICAL PROFESSION—										
Physicians	2,063	52	1·26	1,085	15	·69	978	37	1·89	1·29
Surgeons, Apothecaries	10,189	381	1·87	5,564	155	1·39	4,625	226	2·44	1·91

HOME AND FOREIGN INTELLIGENCE.

ENGLISH AND SCOTTISH LAW LIFE ASSURANCE ASSOCIATION.

Established 1839.

FIFTH DIVISION OF PROFITS AS AT CHRISTMAS, 1870.

The Valuation of the Policies has been made by the Actuary of the Association; and is based on the following principles, viz.:—

1. For Policies of Assurance. The Carlisle Table of Mortality combined with Three Per Cent. Interest, taking into account only the "Pure" or "Net" Premiums, without encroaching on the "Loading" or additions originally made thereto to provide for future Expenses and Profits:—the Present Value of the whole "Loading" thus reserved being £247,378.

2. For Annuities. The Mortality Experience of the Government Annuitants combined with Three Per Cent. Interest.

From the Actuary's Report and Valuation Balance-Sheet hereto annexed, it appears that the Assets and Liabilities of the Office as at Christmas, 1870, estimated on these principles, stood as follows:—

Total Assets, including the Present Value of the Net Premiums	£1,881,029
Total Liabilities, including the Present Value of the Sums Assured and previous Bonus Additions . .	1,803,961
Leaving a Free Divisible Surplus of . .	£77,068

without encroaching in any way on the Reserve for future Expenses and Profits.

Division of Profits.

In accordance with the Provisions of the Deed of Constitution, the declared Surplus was appropriated in manner following:—

1. To the Assured:—Ninety per Cent. of the Divisible Surplus, to provide for the following Reversionary Bonus Additions to all policies entitled to participate in Profits, viz.:—

(1.) An Addition to the Whole Term Policies entitled to participate in Profits, and in force at the time of Division, of a Reversionary Bonus at the rate of One-and-a-Half per Cent. per Annum on the Sum Assured for each Full Annual Premium paid since the previous Division of Profits.

(2.) An Addition to the Endowment-Assurance Policies, entitled to participate in Profits, and in force at the time of the Division, of a Reversionary Bonus at the rate of One per Cent. per Annum on the Sum Assured for each Full Annual Premium paid since the previous Division of Profits.

The Total Amount of these Reversionary Bonus Additions is £116,600; and the Present value thereof	£67,331
2. To the Proprietors:—Ten per Cent. of the Divisible Surplus, as their proportion of Profits to be applied by way of Increase to the Ordinary Dividend during the next Five Years .	7,706
Making a Total of	£75,537
And leaving a Balance to be reserved of.	1,531
Total	£77,068

Appropriation of Bonus.

The Bonus is declared in the first instance as a Reversionary Addition to the Sum Assured; but it may, at the option of the Assured, be at once converted into an Immediate Cash Payment, or into a Reduction of the Annual Premium either during the next Five Years or during the Remainder of Life. Thus, a Policy for £2,000 opened in 1840 on a life then aged 35, at an Annual Premium of £56. 10*s*., on which the Vested Bonus Additions now amount to £1,000, may remain on the books as an Assurance of £3,000; or it may be discharged of all Premiums for the next Five Years on a surrender of £422 of the Bonus, leaving the Sum Assured £2,578; or it may be discharged of all Premiums during the Remainder of Life on a surrender of £870 of the Bonus, leaving the Sum Assured £2,130 subject to no further payments; or the whole Bonus may be converted into an Immediate Cash Payment of £577. 10*s*. In all these cases the Policy remains entitled to future Bonus Additions, which, at each recurring period of Division, may again be appropriated in accordance with the wishes of the Assured.

BONUS ADDITIONS TO POLICIES OF £1,000 EFFECTED SINCE 1840.

Policy opened in	Original Sums Assured.	Bonus Additions.	At Christmas, 1870.
	£	£	£
1840	1,000	500	1,500
1845	1,000	390	1,390
1850	1,000	315	1,315
1855	1,000	240	1,240
1860	1,000	165	1,165
1865	1,000	90	1,090
1870	1,000	15	1,015

These Bonuses are not deferred, but vest immediately on Declaration.

VALUATION BALANCE SHEET:—CHRISTMAS, 1870.

LIABILITIES.

Dr.		£ s. d.	£ s. d.
To Present Value of all Sums Assured, and Past Bonus, after deducting Sums Re-Assured (£3,229,784)			1,656,661 9 0
„ Capital paid up	£40,000 0 0		
„ Bonus Additions thereto	30,000 0 0		
		70,000 0 0	
„ Proprietors' Reserve Fund		2,945 4 3	
„ Present Value of Life Annuities (£6,680)		40,564 4 7	
„ Claims admitted, but not paid.		30,215 0 0	
„ Dividends due		2,895 16 0	
„ Annuities and Salaries due		680 0 0	
			147,300 4 10
„ Surplus, viz.:—			
To the *Proprietors*, One Tenth of the Profits to be applied by way of Increase to the ordinary Dividend during the next Five Years		7,706 0 0	1,803,961 13 10
To the Assured, to provide for Reversionary Bonus Additions (Total, £116,600) to all Policies entitled to participate		67,831 0 0	
		75,537 0 0	
Balance carried forward		1,530 19 6	
			77,067 19 6
			£1,881,029 13 4

ASSETS.

Cr.

	Amount.	Value.					
By Present Value of future Net Premiums, viz.:—	£	£	*s.*	*d.*	£	*s.*	*d.*
Office Yearly Premiums	97,433	1,368,275	2	2			
Less "Loading" reserved	19,097	247,378	5	9			
Net Yearly Premiums	78,336	1,120,896	16	5			
					1,120,896	16	5
„ Assets, as per Balance Sheet presented to the Annual Meeting on the 22nd February, 1871, and certified by the Auditors, viz.:—							
Mortgages on Property within the United Kingdom		278,114	3	2			
Loans on the Company's Policies		21,363	10	5			
Investments—							
In British Government Securities		47,806	6	1			
„ Indian and Colonial Government Securities		91,998	15	3			
„ Railway and other Debentures and Debenture Stocks		1,560	0	0			
„ Railway Shares (Preference)		25,681	18	7			
„ House Property		26,518	7	0			
Loans on Personal Security		188,149	6	2			
Agents' Balances		666	15	3			
Outstanding Premiums		8,246	18	5			
Do. Interest		11,347	10	3			
Cash—							
On Deposit ..£43,000 0 0							
In hand and on current account...... 15,584 19 4							
		58,584	19	4			
Other Assets—							
Stamps on hand		94	7	0			
					760,132	16	11
					£1,881,029	13	4

The following further particulars are extracted from the Returns under the Fifth and Sixth Schedules of the Life Assurance Companies' Act.

The principles upon which the Valuation is made are in the discretion of the Board of Directors and the Actuary, and the Distribution of the Profits is regulated by the Deed of Constitution and the Resolutions endorsed thereon, which provide that the profits arising from the Life Assurance Fund shall be appropriated by giving One-Tenth to the Proprietors, to be applied by way of increase to the ordinary Dividend during the next succeeding five years, and Nine-Tenths to be applied as a Reversionary Bonus to those Policies entitled to participate in profits.

For Policies of Assurance,—the Carlisle Table of Mortality. For Annuities,—the Mortality Table of the Government Annuitants.

The Rate of Interest assumed in the calculations is Three per Cent.

The whole of the "loading" added to the pure or net Premium to provide for future expenses and profits, is reserved for that purpose.

All Policies effected on the Participating Scale under full Premiums for the Whole Term of Life, and also all Endowment Assurance Policies effected on the Participating Scale, and in force at the time of the Division, are entitled to share in profits.

The Amount of Profit Divided on this occasion among the Policy-holders is £67,831, among 3,060 Policies, for the original sum assured of £1,853,412 0*s.* 0*d.*

Specimens of Reversionary Bonuses allotted at the Division for the Five Years ending 25th December 1870, to Policies for £100, effected at the Ages of 20, 30, 40, 50, are subjoined.

Age at Entry.	POLICIES OF £100 IN FORCE FOR					
	5 Years.	10 Years.	15 Years.	20 Years.	25 Years.	30 Years.
	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*
20	7 10 0	7 10 0	7 10 0	7 10 0	7 10 0	7 10 0
30	7 10 0	7 10 0	7 10 0	7 10 0	7 10 0	7 10 0
40	7 10 0	7 10 0	7 10 0	7 10 0	7 10 0	7 10 0
50	7 10 0	7 10 0	7 10 0	7 10 0	7 10 0	7 10 0

The Average Rate of Interest at which the Life Assurance Fund was invested during the period since the last investigation, has been,

For the year 1866, at £4 9*s.* 4*d.* per cent. ;
" 1867, at £4 8*s.* 10*d.* "
" 1868, at £5 0*s.* 11*d.* "
" 1869, at £4 13*s.* 2*d.* "
" 1870, at £4 11*s.* 1*d.* "

Surrender values of POLICIES for £100, exclusive of the value of any Bonus added thereto.

Age at Entry.	DURATION OF THE POLICY.			
	5 Years.	10 Years.	15 Years.	20 Years.
	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*
20	2 14 5	5 13 0	8 12 6	12 0 4
30	3 5 6	7 1 0	10 18 8	15 6 8
40	4 4 7	9 7 10	15 13 1	22 0 0
50	7 8 6	14 18 10	21 2 3	28 2 10

Consolidated Revenue Account for Five Years, commencing 25th December 1865, *and ending 25th December* 1870.

	£	s.	d.		£	s.	d.
Amount of Funds on 25th December 1865, the beginning of the period	598,297	2	1	Claims under Policies (after deduction of Sums Re-assured)	372,947	11	11
Premiums (after deduction of Re-assurance Premiums) ..	464,864	14	4	Surrenders	28,772	4	2
Consideration for Annuities granted	12,904	11	0	Annuities	30,970	5	7
Interest and Dividends	161,813	12	10	Commission	20,096	4	2
Other Receipts :—				Expenses of Management..	49,623	10	9
Reversions	9,029	14	7	Dividends, and Bonus to Shareholders	25,000	0	0
Balance of Profit and Loss	2,945	4	3	Amount of Funds on 25th December 1870, the end of the period, as per First Schedule	726,342	0	11
Increased Value of Real Property	3,556	9	10				
Registration Fees	110	7	0				
Revival Fines	230	1	7				
	£1,253,751	17	6		£1,253,751	17	6

Valuation Balance Sheet, as at 25th December 1870.

Dr.	£ s. d.	Cr.	£ s. d.
To Net Liability under Assurance and Annuity Transactions (as per Summary Statement provided in Schedule 5)	576,328 17 2	By Life Assurance and Annuity Funds (as per Balance Sheet under Schedule 2) ..	653,396 16 8
To Surplus	77,067 19 6		
	£653,396 16 8		£653,396 16 8

Summary and Valuation, as at 25th December 1870.

Description of Transactions.	Particulars of the Policies for Valuation. Number of Policies.	Sum Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Valuation. Value by the Carlisle Table of Mortality with Interest 3 per Cent. for Assurances, and by the Government Annuitants' Experience with Interest 3 per Cent. for Annuities. Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Net Liability.
ASSURANCES.								
I. *With Participation in Profits.*								
For Whole Term	2,811	1930396·90	54560·40	42888·64	1057086·63	787573·29	617097·18	439989·45
Limited Payments	64	32636·00	1451·46	1164·43	15435·58	11331·70	9090·83	6344·75
Ascending Scale	1	1075·00	32·90	26·32	538·65	547·61	437·76	100·95
Endowment Assurances	160	49389·50	1994·00	1629·67	29216·79	24150·80	19738·14	9478·65
Joint Lives	24	10975·50	421·56	331·87	6662·17	557·47	4388·69	2273·48
Extra Premiums payable	—	— —	844·29	— —	844·29	— —	— —	844·29
Total Assurances with Profits	3,060	2,024,472·90	59,304·61	46,040·93	1,109,784·11	824,160·87	650,752·54	459,031·57
II. *Without Participation in Profits.*								
For Whole Term	1,584	1430840·	46730·49	40186·47	732982·48	691890·40	592992·71	139989·77
Limited Payments	16	8950·	425·85	372·42	4847·74	2612·09	2284·36	2563·38
Ascending Scale	6	11000·	314·06	273·20	5707·48	4934·01	4292·08	1415·40
Endowment Assurances	31	8961·	373·92	332·16	5659·74	4571·44	4060·88	1598·86
Joint Lives	14	16050·	818·35	671·30	9846·27	10636·80	8725·53	1120·74
Last Survivor	12	11250·	199·90	161·82	4068·14	3837·63	3106·58	961·56
Term Assurances	49	92212·	1920·00	1344·00	3417·00	2880·01	2016·00	1401·00
Contingent Assurances	61	128204·	962·66	732·71	9893·57	7720·00	5875·92	4017·65
Children's Endowments	155	25900·	1065·86	1003·86	17326·65	9047·78	8521·44	8805·21
Extra Premiums payable	—	— —	855·95	— —	855·95	— —	— —	855·95
Total Assurances without Profits	1,928	1,733,387·	53,667·04	45,077·94	794,605·02	738,130·16	631,875·50	162,729·52
Total Assurances	4,988	3,757,859·90	112,971·65	91,118·87	1,904,389·13	1,562,291·03	1,282,628·04	621,761·09
Deduct Re-assurances	—	528,075·72	15,448·18 }	12,782·79	247,637·68 }	194,015·92	161,731·22	85,906·46 }
" Extra Premiums payable			90·00 }		90·00 }			90·00 }
Net amount of Assurances	—	3,229,784·18	97,433·47	78,336·08	1,656,661·45	1,368,275·11	1,120,896·82	535,764·63
ANNUITIES.								
Immediate	108	5797·85	— —	— —	38924·18	— —	— —	38924·18
Contingent Survivorship	7	1895·00	487·25	414·22	4245·71	3541·91	3011·04	1234·67
Deferred	1	40·00	7·00	6·30	134·00	94·22	84·80	49·20
For Term of Years		46·83	— —	— —	725·78	— —	— —	725·78
Total Annuities	117	7,779·68	494·25	420·52	44,029·67	3,636·13	3,095·84	40933·83
Deduct Re-assurances	—	1,100·00	— —	— —	369·60	— —	— —	369·60
Net amount of Annuities	—	6,679·68	494·25	420·52	43,660·07	3,636·13	3,095·84	40,564·23
Total of the Results	—	— —	97,927·72	78,756·60	1,700,321·52	1,371,911·24	,123,992·66	576,328·86

NOTE.—The extra premiums are for temporary risks incurred by naval and military service, voyages and residence in foreign countries beyond the limits of Europe.

LONDON AND PROVINCIAL LAW ASSURANCE SOCIETY.

Established 1845.

Bonus 1870. Fourth Division of Profits.

REPORT FROM THE DIRECTORS.

Another quinquennial period having been completed on the 31st December 1870, the Directors have instituted a careful investigation into the Society's affairs, with the view of ascertaining the amount of profit which can be safely set apart out of the Assurance Fund for division between the Proprietors and the Assured; and they have now the pleasure to report the results.

The Society has now been upwards of twenty-five years in existence, and has issued 3,543 Policies for the amount of £4,314,980. 18*s.* 3*d.*, of which 2,107 Policies, assuring £2,847,449. 3*s.* 3*d.*, now remain in force. The existing assurances show an increase of 48 per cent. in amount over those in force on the 31st December, 1865.

The total claims by death have amounted to £332,909. 18*s.* 4*d.*, of which £153,436. 13*s.* 2*d.* were paid or provided for in the quinquennial period just concluded. The claims have been within the expectation.

The Investments have yielded an average rate of interest of £4. 15*s.* 3*d.* per cent. per annum.

The items composing the general income and outgo of the Society during the Quinquennium will be seen in the appended "Consolidated Revenue Account," which is made up in accordance with "The Life Assurance Companies Act, 1870," 33 and 34 Vic., cap. 61.

On the 31st December,	1865,	the total Funds were	£383,612	2	8
Do.	1870,	do.	618,405	18	3

Thus showing an increase in the Quinquennium of £234,793. 15*s.* 7*d.*, which is at the rate of 63 per cent.

It will be seen by the Balance Sheet that the Assurance Fund amounts to £520,721. 19*s.* 9*d.*; and in the same account are detailed the several securities in which it and the Proprietors' Fund are invested.

These Securities are all of a high class, and yield at the present time an average rate of interest of £4. 16*s.* 8*d.* per cent. per annum.

The Reversions and other Investments have been very carefully valued.

BALANCE SHEET on 31st DECEMBER, 1870.

Dr. Liabilities.				£	s.	d.
Shareholders' Capital paid up	£97,683	18	6			
Assurance Fund	520,721	19	9			
Total Funds				618,405	18	3
Claims admitted but not paid				6,093	0	0
Annuities due				91	15	5
Dividends unpaid				406	0	10
				£624,996	14	6

Cr. ASSETS.	£	s.	d.
Mortgages on Property within the United Kingdom ..	367,548	19	3
Mortgage on Property out of the United Kingdom. On Indian Government Stock	3,500	0	0
Loans on the Company's Policies	5,717	13	10
Investments:—			
In British Government Securities	36,030	0	0
Foreign Government Securities	19,952	0	0
Railway and other Debentures	48,086	2	6
Railway Shares (Preference)	7,265	9	1
House Property—Society's House, 21, Fleet Street	4,800	0	0
Indian Railway Guaranteed Stock	46,224	0	0
Reversions valued at	46,375	10	4
Life Interests valued at	12,312	19	8
Loans upon Personal Security	16,353	12	0
Outstanding Premiums	5,035	11	3
Ditto Interest	1,121	15	7
Cash:—			
In hand and on current Account	4,673	1	0
	£624,996	14	6

The new business of the past five years shows a very considerable increase over that transacted in the years 1861–1865, as seen in the following Table:—

STATEMENT OF NEW BUSINESS EFFECTED IN THE TWO PAST QUINQUENNIUMS.

	1861–65.												Increase per-cent. in the new Amounts assured in the Quinquennium 1861–65.	
	No. of Pols.	Sum Assured.			Premiums. Annual.			Premiums. Single.			Average of each Policy.			
		£	s.	d.	£	s.	d.	£	s.	d.				
Participating	560	571,440	9	0	19,359	17	2	....			1,020			
Non-partic...	155	198,690	0	0	7,131	7	5	....			1,280			
Contingent ..	31	55,700	0	0	1,164	17	5	52	6	9	1,800			
Term	45	54,370	0	0	1,298	7	8	375	19	9	1,210			
Issue	50	172,716	5	0	....			9,155	17	3	3,454			
Sundries	62	76,880	0	0	3,155	9	6	923	19	10	1,240			
	903	1,129,796	14	0	32,109	19	2	10,508	3	7	1,250			
	1866–70.												Inc.	Dec.
		£	s.	d.	£	s.	d.	£	s.	d.				
Participating	549	703,656	0	0	22,546	13	11	....			1,282		23·	....
Non-partic...	221	401,705	0	0	14,873	7	7	....			1,818		100·	....
Contingent ..	57	111,953	0	0	1,350	0	11	2,302	10	0	1,950		100·	....
Term	60	91,650	0	0	1,573	5	2	622	17	3	1,527		74·	....
Issue	30	94,940	0	0	33	5	0	5,112	14	8	3,164		....	66·
Sundries	53	106,241	14	3	2,464	17	10	2,095	11	0	2,000		40·	....
	970	1,510,145	14	3	42,841	10	5	10,133	12	11	1,556		34·	

A Classified Statement of the Policies in force, with the immediate net liability thereunder, will be found below. This Statement is prepared in the form prescribed by "The Life Assurance Companies Act, 1870."

The following Table (on p. 137) shows that the immediate net liability of the Society under its several Assurance and Annuity transactions is £412,476.

In ascertaining this liability, the **net** premiums only have been valued, leaving the whole of the "loading" on the premiums as a provision for future profits and expenses.

The Tables of Mortality used are—

1. For the Assurance Contracts, "The 17 Offices' Experience," which Table shows the rate of mortality actually prevailing amongst Assured Lives, and necessitates the highest reserve of all known tables.
2. For the Annuity Contracts, "Davies' Equitable" Table, which, for such purpose, is appropriate, and also requires a high reserve.

The rate of interest adopted in both cases is 3 per cent. only.

Adding to the net liability of £412,476 so found, £5,054 for claims announced in the year and standing for admission, and a further sum of £500 for contingencies, the Valuation Reserve becomes £418,030.

The surplus on the Assurance Fund is therefore £102,691. 19*s*. 9*d*. Of this sum the Directors recommend that £100,000 be divided on this occasion, leaving the balance, £2,691. 19*s*. 9*d*., to accumulate for future division.

One-fifth, or £20,000, of the sum now appropriated will be carried to the Proprietors' Fund, (in accordance with the provisions of the Deed of Settlement,) equivalent to an addition of £1 per Share. The amount paid up on each Share will therefore stand at £5. 17*s*. 8*d*.

The remaining £80,000 will be appropriated to the Assured, and will be divided amongst those entitled to participate, according to the same principles as have been heretofore adopted. The Bonus attaching to each Policy, with the alternative commutations, will be at once communicated to the Assured.

The general results of the several divisions of profits are shown in the following Table:—

TABLE OF BONUS ADDITIONS

Attaching to Policies of £1,000 each.

Age at Entry.	NUMBER OF PREMIUMS PAID.									
	TWENTY-FIVE.		TWENTY.		FIFTEEN.		TEN.		FIVE.	
	Previous Bonuses.	Bonus 1870.	Previous Bonuses.	Bonus 1870.	Previous Bonuses.	Bonus 1870.	Previous Bonuses.	Bonus 1870.	Previous Bonuses.	Bonus 1870.
	£	£	£	£	£	£	£	£	£	£
20	314	79	240	78	169	78	81	73	—	73
30	336	86	258	84	180	83	90	81	—	80
40	375	98	287	92	199	94	100	90	—	88
50	443	134	331	116	225	110	119	104	—	104

* * * * *

SUMMARY AND VALUATION OF THE POLICIES, AS AT 31st DECEMBER, 1870.

DESCRIPTION OF TRANSACTIONS.	PARTICULARS OF THE POLICIES FOR VALUATION.				VALUATION. Assurances Valued by "17 Offices Experience" Table. Annuities Valued by "Davies' Equitable" Table. Interest 3 per Cent.			
	No. of Policies	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Prems.	Sums Assrd. and Bonuses.	Office Yearly Prems.	Net Yearly Prems.	Net Liability.
ASSURANCES.								
I.—With Participation in Profits.		£ s. d.	£ s. d.	£	£	£	£	£
For the whole term of life	1,425	1,726,126 9 0	51,763 12 0	41,626	912,257	763,218	606,988	
Less reductions of Premium by way of Bonus	....	....	136 6 9	136	....	1,529	1,529	
			51,627 5 3	41,490	....	761,689	605,459	306,798
Extra Premiums thereon	....	....	713 4 0	..	357	....	..	357
Other Classes—								
Last Survivor	6	14,524 0 0	229 3 7	163	6,124	4,272	3,052	3,072
Limited and Commuted Premiums	15	29,847 0 0	858 8 5	674	17,561	9,581	7,619	9,942
Ascending and Descending Scales	18	52,527 0 0	1,411 19 8	1,112	24,693	26,953	21,302	3,391
Extra Premiums thereon	....	....	60 0 0	..	30	....	..	30
Sundries	3	11,500 0 0	486 8 9	415	4,025	3,611	3,025	1,000
Total Assurances with Profits	1,467	1,834,524 9 0	55,386 9 8	43,854	965,047	806,106	640,457	324,590
II.—Without Participation in Profits.								
For the whole term of life	418	644,898 0 0	23,195 9 9	20,944	357,773	308,220	275,184	82,589
Extra Premiums thereon	....	....	193 15 0	..	97	....	..	97
Other Classes—								
Contingent	57	119,103 0 0	1,643 1 4	1,198	11,676	10,977	7,766	3,910
Extra Premiums thereon	....	....	5 0 0	..	3	....	..	3
Temporary	31	43,600 0 0	758 19 3	618	1,698	1,677	1,366	332
Joint Lives	14	9,850 0 0	576 3 9	505	6,715	5,618	4,874	1,841
Last Survivor	30	58,310 0 0	1,120 3 9	944	24,462	20,926	17,749	6,713
Endowments and Endowment Assurances	13	6,550 0 0	396 4 5	359	4,850	2,206	1,964	2,886
Limited and Commuted Premiums	5	9,431 0 0	195 0 0	166	3,982	1,966	1,612	2,370
Ascending and Descending Scales	9	17,100 0 0	505 16 6	443	9,311	8,620	7,676	1,635
Sundries	3	10,950 0 0	178 10 0	139	6,678	7,465	6,305	373
Issue, &c.	60	213,350 14 3	....	..	8,771	....	..	8,771
Total Assurances without Profits	640	1,133,142 14 3	28,768 3 9	25,316	436,016	367,675	324,496	111,520
Total Assurances	2,107	2,967,667 3 3	84,154 13 5	69,170	1,401,063	1,173,781	964,953	436,110
Deduct Reassurances	....	342,231 5 5	8,664 17 0	7,234	142,873	124,993	106,180	36,693
Net Amount of Assurances	....	2,625,435 17 10	75,489 16 5	61,936	1,258,190	1,048,788	858,773	399,417
ANNUITIES.		Per Annum.						
Immediate	30	1,637 3 3	....	..	12,215	....	..	12,215
Contingent	2	350 0 0	202 6 11	164	2,080	1,520	1,236	844
Total of the Results	....	....	75,692 3 4	62,100	1,272,485	1,050,308	860,009	412,476

The Board are aware that the present surplus might have been largely increased had they adopted less stringent principles in the valuation of the Society's Liabilities; but they entertain a strong opinion of the impolicy of any lowering of the standard of safety.

* * * * *

CONSOLIDATED REVENUE ACCOUNT

For Five Years, commencing 1st January, 1866, *and ending* 31*st December,* 1870.

Dr.	£	s.	d.
Amount of Funds, 1st January, 1866, the beginning of the Quinquennium	383,612	2	8
Premiums (after deduction of Reassurance Premiums)	349,905	12	11
Consideration for Annuities granted	6,908	11	7
Interest and Dividends	105,480	4	3
Fines for Revival of Lapsed Policies	176	10	8
Consideration for 1,526 of the Society's Shares	7,451	19	4
Reversionary Life Interest fallen into possession—Increase in Value	2,055	6	8
	£855,590	8	1

Cr.	£	s.	d.
Claims under Policies (after deduction of sums Reassured)	155,101	13	2
Surrenders	10,503	19	6
Annuities	6,504	0	0
Commission	16,988	1	0
Expenses of Management	19,782	16	2
Dividends to Shareholders	21,126	12	0
Reversionary Bonuses commuted and paid in Cash	7,177	8	0
Amount of Funds on 31st December, 1870, the end of the Period	618,405	18	3
	£855,590	8	1

[From the Chairman's address it appears that the Government Securities have been taken at 90, whereas the average price during the last twenty years has been 92, and the present price is between 92 and 93. The Great Indian Peninsula Guaranteed Stock has been taken at 106, it now stands at 108; the Great Southern of India at 103, it is now between 105 and 106; and the value of the Society's House is estimated at £4,800, which is confessedly a very low value. The reversions have been taken at the market value. The average rate of interest during the five years has been in 1865, £4. 13*s.* 7*d.* percent; in 1866, £4. 16*s.*; in 1867, £4. 17*s.*; in 1861, £4. 15*s.*; and in 1869, £4. 15*s.*; giving an average of £4. 15*s.* 3*d.* per cent.]

The following further particulars are extracted from the Returns under the Fifth and Sixth Schedules of the Life Assurance Companies' Act.

The valuation was made on the following principles :—

The assurances were valued in classes according to the ages attained by the lives assured on the 31st December, 1870.

The ages attained were taken as the office-ages at entry, plus the number of complete years elapsed since the grant of the assurance. Lives assured at increased rates were assumed to be, at entry, of the ages corresponding to the premiums charged.

The net premiums valued were those arising from the "17 Offices' Experience" Table of Mortality with 3 per cent. interest. Each premium was assumed to fall due half-a-year hence.

The assurances, other than ordinary whole term assurances, and the annuities, were valued singly.

The Profits have been divided among the Policy-holders entitled to participate as follows :—

(*a.*) As to those who shared in the Profits on the last division, in proportion to the premiums they have since paid ; previous Bonus addtion. ranking as new assurances effected at the date of their allotment.

(*b.*) As to all others assured on the participating scale, and who are entitled to share in the present division, in proportion to *all* the premiums they have paid.

The sums so apportioned were converted into equivalent reversionary additions.

The principles to be adopted in the valuation and distribution of the Profits are not determined by the instrument constituting the Company, or by its regulations, or byelaws ; but, by the Deed of Settlement, are left in the discretion of the Board of Directors.

The Tables of Mortality used in the valuation were :—

(*a.*) For the Assurance contracts, the "17 Offices' Experience" Table. In some very few instances, where the subsidiary Tables on this basis were not sufficiently extensive, the "Carlisle" Table was used.

For the Assurances depending on the birth of Issue, a special reserve was made, amounting to three-fourths of the premiums received in respect of the same.

Half-a year's premium was reserved for the current risks under annually renewable licences granted for foreign residence : where a single payment had been made for "whole world" licences, the value of the equivalent annual premium was reserved.

(*b.*) For the Annuity Contracts, the "Davies' Equitable" Table.

The rate of interest assumed in the calculations was 3 per cent. throughout.

The whole of the "Loading" upon the net premiums valued (17 Offices' Experience, 3 per cent.), averaging 20 per cent., has been reserved as a provision for future expenses and profits.

A policy is entitled to share in the Profits when three annual premiums have been paid.

The number of Policies which participated was 1,371, and their amount was £1,675,835. 9s.

The following are specimens of Bonuses allotted to Policies for £100 each, and the amounts apportioned under the various modes in which the same might be received.

Age at Entry.	NUMBER OF YEARS IN FORCE.														
	5			10			15			20			25		
	Reversionary Addition.	Cash.	Permanent Reduction of Premium.	Reversionary Addition.	Cash.	Permanent Reduction of Premium.	Reversionary Addition.	Cash.	Permanent Reduction of Premium.	Reversionary Addition.	Cash.	Permanent Reduction of Premium.	Reversionary Addition.	Cash.	Permanent Reduction of Premium.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	7 0 0	2 1 0	0 3 1	7 0 0	2 5 0	0 3 7	8 0 0	2 13 0	0 4 3	8 0 0	2 19 0	0 4 11	8 0 0	3 6 0	0 5 11
30	8 0 0	2 15 0	0 4 4	8 0 0	3 1 0	0 5 1	8 0 0	3 9 0	0 6 1	8 0 0	3 18 0	0 7 3	9 0 0	4 8 0	0 8 11
40	9 0 0	3 14 0	0 6 5	9 0 0	4 4 0	0 7 10	9 0 0	4 16 0	0 9 9	9 0 0	5 4 0	0 12 1	10 0 0	6 0 0	0 16 6
50	10 0 0	5 8 0	0 10 9	10 0 0	5 18 0	0 13 7	11 0 0	6 16 0	0 18 7	12 0 0	7 16 0	1 5 7	13 0 0	9 15 0	1 18 6

Valuation Balance Sheet, as at 31st December, 1870.

Dr.	£	s.	d.	Cr.	£	s.	d.
To net Liability under Assurance and Annuity transactions (as per Summary Statement provided in Schedule 5)	412,476	0	0	By Life Assurance Fund (as per Balance Sheet under Schedule 2)	520,721	19	9
„ Reserve for Claims announced but not admitted	5,054	0	0				
„ Reserve for Contingent Expenses	500	0	0				
„ Surplus	102,691	19	9				
	£520,721	19	9		£520,721	19	9

The average rate of Interest at which the Life Assurance Fund was invested at the close of each year during the period since the last investigation was—

31st Dec. 1866	£4	16	0	per cent.	per annum.
" 1867	4	17	0	"	"
" 1868	4	15	0	"	"
" 1869	4	15	0	"	"
" 1870	4	16	8	"	"

The Surrender Value of Policies effected for the whole period of life and at uniform annual premiums, and upon which three premiums have been paid, ranges between 33 and 60 per cent. of the total ordinary premiums paid. The Society has no minimum scale or fixed method of calculating the surrender value of other descriptions of risk, but makes a fair allowance based upon the circumstances of each case.

The Society has no business at other than European rates.

CORRESPONDENCE.

MR. SANG'S SEVEN FIGURE LOGARITHMS.

To the Editor of the Assurance Magazine.

SIR,—In reply to my offer of a copy of the new table of seven-place logarithms, for the indication of any error in it, Dr. A. W. Whitcom of Milwaukee, U.S.A., points out a last-place error in the logarithm of 52943. My calculations were made for the numbers above 100,000, and so are not chargeable with this error; its source is in Vlacq's ten-place table (1628), where there is a misprint of 85868 instead of 85468. This has passed into all the shortened tables, it is in Sherwin (1741), Taylor (1792), Callet (1795), Hutton (1804, 1838), Babbage (1841), Shortrede (1844), as also in John Newton's eight-place table (1658), wherein 80859 is put for 80855, thus showing that Vlacq's great work has served for the production of all the others. The true logarithm to seven places is ·7238085; from my manuscript table to 28 places in progress up to 10,000, I find

$$\begin{array}{lrl} \text{Log} & 4813 = & \cdot 68241\ \ 58616\ \ 77358\ \ 49039\ \ 14460\ \ 667 \\ \text{log} & 11 = & \cdot 04139\ \ 26851\ \ 58225\ \ 04075\ \ 01999\ \ 712 \\ \hline \text{log} & 52943 = & \cdot 72380\ \ 85468\ \ 35583\ \ 53114\ \ 16460\ \ 379 \end{array}$$

It is remarkable that this error should have so long escaped detection, and all the more credit is due to its discoverer.

I am, Sir, your obedient Servant,

28 *May* 1872. EDWARD SANG.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.—Session 1871–72.

First Ordinary Meeting, Monday, 27th November 1871.

The President in the Chair.

Read and confirmed the minutes of the anniversary meeting, held on the 3rd June 1871.

The following gentlemen were elected members, viz.:—

Fellows.

William Henry Archer.
Major-General Hannyngton.
Andrew Francis.

Associates.

Peter Ronaldson.
George Leonard Murphy.
William Gordon Glennie.
John Tasker Maitland.
Frank Arthur Straker
Samuel Thomas Skudder.
John Yardley.
Arthur James Cook.
Henry Walsingham Andras.
Hubert George Rowsell.
Henry Cockburn.
James Ritchie Macfadyen.
Edwin Arthur Coutts.
Stanford Smith.
John Sutherland Valentine.
Thomas Henry Whitford.
Tom Ferrers Guy.
Samuel George Robinson.
Niel Ballingal Gunn.
George Rogers Jellicoe.
James Sorley.
Alfred Bell.
Robert James Spencer.
Robert Wilson, Jun.
Francis Ogle Moore, B.A.
Robert Henry Steele.
William Samuel Brown.
Richard Joseph Legg.
James Busset.
Mandeville Blackwood Phillips.
Thomas Homans Cooke.
Francis Laing.
Duncan Brocklebank.
Frederick George Painter.

Mr. A. H. Bailey read a paper "On Insolvency in Life Insurance Companies."

Thanks having been voted to Mr. Bailey, the meeting adjourned to Monday, 18th December 1871.

Second Ordinary Meeting, Monday, 18th December 1871.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz.:—

Samuel Hunter.
Archibald Hewat.
Thomas Charles Dewey.
Joseph Mills.
William Salter.
Roderick Mackenzie Moore.
John Hall Clarke.

Mr. Peter Gray read a paper by Mr. Makeham "On the Laws of Sickness and Invalidism, and their relation to the Law of Mortality."

Thanks having been voted to Mr. Gray and Mr. Makeham, the meeting adjourned to Monday, 29th January 1872.

Third Ordinary Meeting, Monday, 29th January 1872.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentleman was elected an Associate, viz.:—

Leonard Harrison.

The following was announced to be the result of the Examinations for 1871:—

MATRICULATION EXAMINATION.

Thirty gentlemen presented themselves for this Examination, of whom three withdrew, and fourteen passed in the following order of merit:—

1. F. Laing.
2. J. Sorley.
3. S. Hunter.
4. F. A. Straker.
æq. { T. G. Ackland. / C. J. Harvey.
7. H. E. Wilson.
8. H. W. Andras.
æq. { H. W. Eaton. / J. Mills.
æq. { E. Litchfield. / J. McGowan. / C. E. Mason. / V. G. Webb.

SECOND YEAR'S EXAMINATION.

Thirteen gentlemen presented themselves for this Examination, and six passed in the following order of merit, viz.:—

1. C. D. Higham.
2. G. S. Crisford.
æq. { W. T. Gray. / A. Smither.
5. G. King.
6. W. Kember.

THIRD YEAR'S EXAMINATION.

One gentleman presented himself for this Examination, and did not pass.

The best thanks of the meeting were given to the Examiners for their recent services.

Mr. C. J. Bunyon, M.A., read a paper "On the Valuation of Claims upon Current Policies in the Liquidation of a Life Office with reference to the decisions in Bell's and Lancaster's Cases."

Thanks having been voted to Mr. Bunyon, the meeting adjourned to 26th February 1872.

Fourth Ordinary Meeting, Monday, 26th February 1872.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentleman was elected an Associate, viz.:—

William Vaughan.

Mr. W. A. Bowser read a paper entitled "Observations on the Rate of Mortality in Infancy and Childhood."

Thanks having been voted to Mr. Bowser, the meeting adjourned to Monday, 25th March 1872.

Fifth Ordinary Meeting, Monday, 25th March 1872.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentleman was elected an Associate, viz.:—

George Fuller.

Mr. James R. Macfadyen read a paper "On Extra Premium."

Thanks having been voted to Mr. Macfadyen, the meeting adjourned to Monday, 29th April 1872.

Sixth Ordinary Meeting, Monday, 29th April 1872.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

Mr. F. G. P. Neison read a paper "On the Influence of Occupation upon Health, as shown by the Mortality experienced."

Thanks having been voted to Mr. Neison, the meeting adjourned to Monday, 25th November 1872.

The Twenty-fourth Annual General Meeting, Saturday, 1st June 1872.

W. B. Hodge, Esq., the President, in the Chair.

Mr. E. A. Newton (Hon. Secretary) having read the Circular calling the meeting, and the minutes of the last ordinary meeting, read the Report of the Council and the Statement of Accounts, which were as follows:—

"The Council have the satisfaction of reporting an increase in the number " of the members of the Institute.

"Two Fellows and 46 Associates were elected during the year; and the " total number on the books on the 31st March last, after allowing for deaths " and withdrawals, was 296, of whom 105 were Fellows and 191 Associates. " The total number at the close of the preceding year was 271.

"The income and expenditure of the Institute are stated in the accompany- " ing Account, which is duly certified by the Auditors. The ordinary expenses " have amounted to £489 19*s.* 8*d.*, showing a decrease on the previous year; " while the annual subscriptions have reached £596 8*s.*, giving an increase of " £33 12*s.* The several funds available for the purposes of the Institute " amount in all to £2,153 9*s.* 2*d.*, exclusive of the value of the important " collection of books in the Library. The "General Fund" now stands at " £1,151 13*s.* 1*d.*, and has increased by £154 17*s.* 4*d.* in the year.

"The following papers have been read during the past year:—

"*27th Nov.* 1871.—'On Insolvency in Life Assurance Companies.' By Mr. A. H. Bailey.

"*18th Dec.* 1871.—'On the Laws of Sickness and Invalidism, and their relation to the Law of Mortality.' By Mr. W. M. Makeham.

"*29th Jan.* 1872.—'On the Valuation of Claims upon current Policies in the liquidation of a Life Office with reference to the decisions in Bell's and Lancaster's Cases.' By Mr. C. J. Bunyon, M.A.

"*26th Feb.* 1872.—'Observations on the rate of Mortality in Infancy and Childhood.' By Mr. W. A. Bowser.

"*25th Mar.* 1872.—'On Extra Premium.' By Mr. J. R. Macfadyen.

"*29th April* 1872.—'On the Influence of Occupation upon Health, as shown by the Mortality experienced.' By Mr. F. G. P. Neison.

"The Council think it a matter of congratulation that three of these papers "are by new contributors.

"As mentioned in the last Report, the Council have had under considera- "tion the best means of rendering assistance to the younger members of the "Institute in their professional studies, and they have now to report that they "decided upon making the experiment of forming a class for the special study "of the second year's examination subjects, to be conducted by a tutor, who, "being paid in part by the Institute, should, in addition to his class-teaching, "deliver three lectures to be open to all members. Mr. Wm. Sutton, B.A., "was appointed the tutor, and, in consequence of the above arrangements, "students were enabled to avail themselves of all the advantages of his class "at the moderate fee of £2 2*s*. each. The three lectures which were delivered "in conformity with the regulation laid down by the Council, have been "printed in the Journal. Notwithstanding that a period of only three months "intervened between the date at which the class was formed and that of the "examination, the Council have the satisfaction to report that, of ten gentle- "men who joined the class, eight were enabled to present themselves for "examination, of whom six passed. This proof of the advantages which such "a class confers upon the younger members, encouraged the Council to repeat "the experiment; and they are gratified at being able to add that the class for "the present year shows a very considerable increase in the number of "members.

"The Council have the satisfaction to report the completion and publica- "tion of the monetary tables, based upon the Mortality Experience collected "by the Institute. The volume is now on sale, each member of the Institute "being entitled to obtain one copy at half-price at the rooms of the Institute. "To the Tables is prefixed an Introduction by Mr. Peter Gray, in which he "describes at length the processes adopted in their formation, and illustrates "them by detailed numerical examples. The Council take this opportunity "of recording their sense of the important and disinterested services which "Mr. Gray has rendered in the production of the volume, and heartily recom- "mend to all students a careful study of the ingenious processes of calculation "which he has elaborated.

"Appended to the volume is a complete system of Notation for Life Con- "tingencies, which has been settled by a Special Committee with the help of "suggestions and contributions from various quarters; but the Council feel "that they are principally indebted to Mr. Sprague for the great care he gave "to the work of collating and finally bringing into shape the numerous "systems proposed or already in use. The Council entertain a confident hope "that this system will be found well adapted for practical use; and that it "will before long be universally recognized and employed by the actuaries of "all countries as a convenient and efficient medium of intercommunication."

The President said—"Gentlemen, it is now my duty to propose to you, 'That the report of the Council, the abstract of income and expenditure, and the balance sheet, be adopted, entered on the minutes, and printed in the *Journal.*' I think that every one interested in the success of the Institute of Actuaries must allow that the Council are able to present a very satisfactory report, and one which gives very flattering evidence of progress. (Hear, hear.) We are increasing the number of our members, we are increasing our income, and, so far as our ordinary expenditure is concerned, it is somewhat diminished. That, however, must be taken with some allowance, because the extra expenditure for the new class brings us pretty near the ordinary amount. We have had some very important and interesting papers read to us in the course of the session. Our meetings have been very numerously attended, and the papers which we have had the pleasure of hearing at those meetings have given rise to very extensive and valuable discussions—all of which must be very gratifying to every well-wisher of the Institute. (Hear, hear.) The great feature of our last year's proceedings is undoubtedly to be found in the plan which we announced in our last year's report, but about which we were not then in a

position to give definite information. We then stated that we had in contemplation a mode of making the Institute available for the education and assistance of the younger members. That plan was very speedily developed in the form of Mr. Sutton's class, which is particularly referred to in this report; and I think that everybody will now agree that in this matter we have made a great step in forwarding the objects for which the Institute was established. (Hear, hear.) The result has been exceedingly satisfactory. The junior members of the Institute have been able to obtain, at moderate cost, instruction, which they could not have obtained at anything like that cost by any other mode. They have also had three valuable lectures reported in our *Journal*, which will be most useful as matters of reference to the members, who are prosecuting their studies in actuarial science. All this must be very gratifying to every well-wisher of the Institute. Another most important feature, and one which I think is a subject for congratulation, is the publication of this volume which I hold in my hand. (Hear, hear.) We have been engaged—and most laboriously engaged—for a long period in producing tables founded upon the observations of mortality, obtained through the liberality of the various insurance offices. The task has been an exceedingly tedious and heavy one, and we certainly have not been able to bring it to a conclusion so soon as we expected; but I think it will be considered that that conclusion is a most satisfactory one, and that this volume will be found most useful and valuable to every one connected with actuarial science. (Hear, hear.) We propose to sell it to the public at the price of one guinea; every member of the Institute, however, will be entitled to buy a single copy at half price, that is to say, for half-a-guinea, by application at the rooms of the Institute, but of course we can only sell a single copy to each member at that price. This useful book is preceded by a very valuable introduction by Mr. Peter Gray, as to the mode in which the tables were constructed, which is in itself a most important study to every one engaged in actuarial science. I can remember a great many years ago, when I first began to look into these matters, being told by an able actuary and an eminent mathematician, that the theory of the science was complete, that nothing more could be done for it, and that all that remained to be done was to make accurate observations and apply them to the scientific formulæ then established. There is no doubt that the gentleman who made that observation to me was a man of great ability and experience, but his prophecy has been singularly falsified. Since that time, we have had the development of Barrett's method by Mr. Griffith Davies, and we have had the development of Mr. Gompertz's theory by Mr. Makeham and Mr. Woolhouse, which promises to open up a wide field for us in improving our calculations; and we have now got this most important and valuable contribution on the construction of tables by Mr. Peter Gray. It is impossible for me—and in saying this I am sure the Council will agree with me, it is impossible for me to speak in too high praise of the patience and labour that that gentleman has shown in assisting the Council in bringing forward these tables. We owe a deep debt of gratitude to him which I am sure we shall on all occasions be most anxious to acknowledge. (Hear.) I may mention, with regard to the tables, that we are enabled to give, with respect to two lives, what has never been published,—that is, we not only give annuities upon two joint lives, but also annuities upon the last survivor of two lives. It is true that these annuities can be deduced from the tables of annuities ordinarily published, but obtaining them by a tabular reference saves time and labour and the chances of error, and I have no doubt this part of the tables will be found very valuable. (Hear, hear.) The volume also contains a new system of notation. This, I may observe, is what has principally delayed the work, which we should otherwise have been able to publish much sooner. But, as we had long had in contemplation the object of establishing a new system of notation, the Council thought it desirable that it should go forth in this volume, which would make it more generally known than if it were put forth in any other form. For this system of notation we are indebted to several contributors. A meeting of gentlemen connected with actuarial science, not confined to members of the Institute, was called to con-

sider the question of notation. A committee of these gentlemen was formed, and they gave very laborious and constant attention to the consideration of this system. As the report says, we are particularly indebted to Mr. Sprague for the very great and laborious part which he took in the matter. (Hear, hear.) Mr. Sprague laid down a system of notation, and had it printed and circulated amongst the members of the committee. Mr. Samuel Brown also favoured us with some important and valuable suggestions upon Mr. Sprague's notation, which he also printed and circulated; and we had from Mr. F. Allan Curtis, who took a considerable amount of trouble in the matter, a valuable series of suggestions upon the subject, which were printed and considered by the committee. These gentlemen are all members of the Institute. But besides them, we are greatly indebted to Mr. F. Hendriks, who is not a member of the Institute, but who was most constant in his attendance on the committee, and was very anxious indeed to assist in developing a system tending to the general advantage of the profession. (Hear, hear.) These are the principal points connected with the past year that I wish to bring before you, except that I may mention as a gratifying fact that we have had contributions to our proceedings from several young members. Those of us who have been a long time members of the Institute, and engaged in the actuarial profession, have always urged upon the younger members that it is to them we must look for the continuity and permanence of the Institute. It is upon their contributions and their abilities and scientific knowledge that the Institute must rely for its permanent existence, because of course we all grow older and must pass away in time. Therefore, I think it is very gratifying indeed to find that a number of young contributors to our transactions have come forward in the course of last session. (Hear, hear.) I am sure that every one who has observed the proceedings of the Institute, as I have observed them, will feel that there has always been great anxiety on the part of the Council to give every possible facility and encouragement to the younger members, and to secure to them every advantage the Institute can afford them. (Hear, hear.) Gentlemen, there is another point which does not come specially within the business of the meeting, but which perhaps may be gratifying if I mention it to you, and that is that in the course of last session we have been referred to by the Board of Trade as an authority on actuarial matters. I think it gives the Institute a standing and position to be referred to by a great public department on the subject of life assurance, and it will be gratifying to the members to know that such is the case. We were applied to by the Board of Trade for an opinion as to the accounts of a particular company, and we replied that the Institute was strictly a scientific body, and made it a rule not to discuss the affairs of any individual life assurance company. The Board of Trade then put to us some more general queries, irrespective of any particular Office, and asked for information upon the subject. The Council took the matter fully into consideration, and gave answers to the enquiries submitted to them, and this is a letter of acknowledgment which we received from the Board of Trade:—

'Board of Trade, Railway Department,
'London, S.W., 19th May, 1872.

'Sir,—I am directed by the Board of Trade to acknowledge the receipt of your reply of the 15th inst., to the letters from this department of the 2nd inst. and 27th ult., stating the opinion of the Council of the Institute of Actuaries as to what items should be included under the head of "expenses of management," in the accounts of life assurance companies, deposited with the Board of Trade under the Life Assurance Companies Act of 1870.

'I am to request that you will have the goodness to convey the thanks of the Board of Trade to the Council of the Institute of Actuaries for their opinion in the matter.

'I have the honour to be, Sir,
'Your most obedient Servant
(Signed) 'T. GRAY.

'The President of the Council
of the Institute of Actuaries,
'12, St. James's Square, S.W.'

I believe, gentlemen, I have gone through all the matters in the report, and I have now only a few observations to make in consequence of this being the last time I shall have the honour to preside over your meetings as President of the Institute. Originally, you will recollect, we had a permanent President. We went on very satisfactorily under our first two Presidents, Mr. Finlaison and Mr. Jellicoe, and the custom was to re-elect the President every year. But Mr. Jellicoe felt that that was not a satisfactory arrangement, and he strongly urged upon the Council the necessity of a change. We fell in with his view, however much we regretted his leaving the chair; and we then came to a sort of understanding that we would elect a new President every three years. I should have felt great pleasure and gratification in holding for another year the distinguished position in which you were kind enough to place me two years ago, if I felt I was in a position to perform the duties satisfactorily, and to do justice, so far as my humble abilities extend, to the office you so kindly conferred upon me. But, unfortunately, domestic anxiety has placed me in such a position that I feel it incumbent upon me to retire; and I therefore thought it right to give the Council an early intimation to that effect. The circumstances that have induced me to make that intimation are unfortunately still more urgent than they were, and I feel that I have made a very proper resolution. I thank you very much for your kindness to me at all times, but more especially since I have been in this chair; and I will now conclude by moving the resolution of which I have already given notice." (Cheers.)

Mr. R. Tucker—"I beg leave to second the adoption of the report."

Mr. Sprague—"Mr. President, I think the statement we have received from you, and the particulars contained in the report, are extremely satisfactory, and it is a matter of congratulation to see that the Institute is pursuing its steady course, increasing in numbers, and in experience and usefulness, and gaining in an increased degree the confidence of the public, and we may add now of the authorities of the Government, which is shown by the letter you have read to us from the Secretary of the Board of Trade. I do not know that you have left much for me to say. You have gone so fully into the whole affairs of the Institute, that if I were to take any particular point and dwell upon it I could not say much more than you have already put before us. Therefore, I will only make one further remark, and it is this: seeing that the funds of the Institute are now increasing year by year, it appears to me, as an individual member, that the Institute has become rich enough, and the Council may take it into consideration whether they cannot find some means of disposing of their surplus revenue to the general advantage of the members. One step has already been taken by the formation of a class to study the Second Year's subjects. Probably the Council may consider at an early date whether they cannot extend the same system to the First and Third Years' examinations. (Hear, hear.) The Second Year's was fixed upon as the most important, and it was decided to try the class as a sort of experiment. That experiment has answered very well, and I should think that competent instructors might be found to undertake a similar task for the First and Third Years' examinations. I am sure that this and all other matters which will in any way promote the interests of the Institute will not be lost sight of by the Council." (Hear, hear.)

The motion for the adoption of the report was then carried unanimously.

Mr. W. P. Clirehugh and Mr. Searle having been appointed scrutineers, a ballot was taken for the election of President, Vice-Presidents, Officers, and Council, for the ensuing year, which resulted in the election of the following gentlemen:—

President.

Robert Tucker.

Vice-Presidents.

Arthur H. Bailey.	Archibald Day.
Charles John Bunyon, M.A.	Thomas Bond Sprague, M.A.

Council.

Marcus N. Adler, M.A.	Stewart Helder.
Andrew Baden.	Augustus Hendriks.
Arthur H. Bailey.	William Barwick Hodge.
*George William Berridge.	Charles Jellicoe.
Samuel Brown.	Claude George Laing.
Charles John Bunyon, M.A.	James Meikle.
Edward Cutbush.	Edward A. Newton, M.A.
George Cutcliffe.	William P. Pattison.
Archibald Day.	Henry William Porter, B.A.
Henry Devereux Davenport.	Henry Ambrose Smith.
*David Deuchar.	Thomas Bond Sprague, M.A.
*Joseph John Dymond.	John Stott.
William John Hancock.	James M. Terry.
*Major-Gen. J. C. Hannyngton.	Robert Tucker.
Ralph Price Hardy.	John Hill Williams.

Treasurer.

George Cutcliffe.

Honorary Secretaries.

Ralph Price Hardy. | Edward Algernon Newton, M.A.

The Chairman—"Gentlemen, we are very much indebted to Mr. Clirehugh and Mr. Searle for undertaking the duty which they have just performed, and I think we ought to give them a vote of thanks for their services. If it is your pleasure to give them a vote of thanks, you will please signify your approval in the usual way."

The motion was carried unanimously.

The Chairman—"It is the duty of the meeting to appoint auditors. We have now Mr. Emmens, Mr. Hopkinson, and Mr. Searle, who have acted as auditors for the past year. It is necessary that the auditors should be Associates of the Institute. Mr. Emmens has become a Fellow, and, consequently, he is not qualified to act again. Mr. Hopkinson and Mr. Searle have expressed their willingness to serve again, and Mr. T. E. Young has been suggested as the new auditor. I will, therefore, propose that these gentlemen be the auditors for the ensuing year.

Mr. Clirehugh seconded the motion, which was at once agreed to.

Mr. S. Brown—"Mr. President, I am about to propose a resolution, which under ordinary circumstances, forms part of the more extended resolution, which includes all the officers of the Institute. I think on this occasion we are called upon to pass a special vote of thanks to our late President, and that, therefore, we may fairly separate his name from the rest of the officers, and give him a cordial and hearty vote of thanks for the great services he has rendered to the Institute. (Cheers.) During the time he has been in the chair we have gone on most prosperously, and seem to have accomplished even more than we have ever done in any similar period before. For the great prosperity which is at the present moment attending the Institute, we are greatly indebted to the services of the late President, Mr. Hodge, and, therefore, what I would beg to propose is this, 'That the warmest thanks of this Institute be offered to the retiring President, Mr. W. B. Hodge, for the great services he has rendered to this Institute during his period of office, and for the ability and courtesy with which he has on all occasions presided over its meetings.' (Cheers.)

Mr. R. Tucker seconded the motion, which was received with great cordiality.

The President—"I am sure, gentlemen, I am very deeply indebted to you for taking such a kind and partial view of my endeavours to forward the

* New Members.

interests of the Institute. I cannot entirely accept the flattering compliment which Mr. Brown has been kind enough to pay me, in saying that I have had much more influence in forwarding the interests of the Institute during the last two years than other members of the Council. When the Council did me the honour to recommend me to your notice as President, and you did me the honour of electing me, I certainly should not have undertaken the duties of the office if I had not felt confident that I should be supported by the very able and experienced gentlemen whom I had the pleasure of meeting at the Council. For the assistance I have invariably received from them, and the support they have on all occasions afforded me in the discharge of my duties as President, I beg leave to tender my thanks to the Council. It is gratifying to me to think that the gentleman who is to succeed me is much more calculated to further the interests of the Institute than I could possibly be. Mr. Tucker is a gentleman who is so well known to you that he needs no praise from me, although some of the younger members may not be aware of the great service he rendered at the early establishment of the Institute, and the valuable assistance he afforded during the long period he held the office of honorary secretary. (Hear, hear.) He was also very active as one of the early examiners of the Institute, in which I had the honour of being associated with him, and I do not think we could have committed the interests of the Institute to a gentleman who is better qualified to uphold them, by scientific knowledge, great experience, high character, and the eminent position which he holds in the profession. (Cheers.) Again, gentlemen, I thank you for your kindness, and for the partial view you have taken of my humble services; and although I may not perhaps be able at present to render much assistance to the Institute, I shall at all times, in whatever capacity I may be placed, be anxious for its success, and desirous to forward its interests in every way that may lie in my power. (Cheers.)

Mr. W. A. Bowser—"I have very much pleasure in moving that the best thanks of the meeting be given to the retiring Vice-Presidents, Council, and other officers, for their valuable services during the past year. I am sure that you will all cordially join with me in passing this resolution, which is so thoroughly deserved. As one of the younger members, I can bear testimony to the kind way in which the Council have encouraged the younger members during the past year, and I think it is our duty to acknowledge that kindness on the present occasion. (Hear, hear.) I will, therefore, without further remarks, move the resolution which I have read."

Mr. Mountcastle seconded the resolution, which was carried unanimously.

Mr. A. H. Bailey—"On behalf of the other Vice-Presidents and myself, I thank you very much for the kind manner in which you have appreciated the services we have rendered to the Institute. As you have heard such a full exposition of the affairs of the Institute, I do not think it necessary to add anything on this occasion, except that I have no doubt the Council will take up Mr. Sprague's suggestion, and, as we are richer than we were some years ago, will do what in them lies to apply the income of the Institute to the general benefit of the members, and our younger members in particular. (Hear, hear.) I do not think there is any occasion for us to go on accumulating money, and therefore I hope some means will be found for still further extending the usefulness of the Institute." (Hear, hear.)

The President announced that the library would be closed as usual during the month of September.*

* The above report of the proceedings is extracted from the *Insurance Record.*

Income and Expenditure for the Year ended 31*st March* 1872.

Cr.

72.	£	s.	d.	£	s.	d.
of Funds on 31st March 1871, viz.:—						
rtality Experience Fund	430	14	0			
essenger Legacy ,,	239	1	10			
Hardy Memorial ,,	206	10	11			
Brown Prize ,,	204	18	10			
General ,,	996	15	9			
				2078	1	4
Annual Subscriptions, viz.:—						
69 Town Fellows	217	7	0			
27 Country ,,	56	14	0			
Town Associates	252	0	0			
Country ,, (including one paid in advance)	73	10	0			
				599	11	0
on compounded				31	10	0
Experience sales				30	0	6
e Fee				5	5	0
on Messenger Legacy (£211. 1*s.* 10*d.* Cons.)	6	3	3			
Hardy Memorial (£200 Consols)	5	17	0			
Brown Prize Fund (£95. 15*s.* 5*d.* 10½ per-cent East India Stock)	9	16	0			
£688. 18 …	20	3	3			
[illegible]	24	16	9			
				66	16	3
…nditure on …				1	7	0
				£2,812	11	1

1871–72.	£	s.	d.	£	s.	d.
Rent	100	0	0			
Salaries	129	4	0			
Journal	159	7	0			
Tutor	31	10	0			
Stationery	16	9	0			
Ordinary Meetings	17	9	4			
Subscriptions unrecovered	6	6	0			
General Expenses	67	10	4			
				527	15	8
Mortality Experience Expenses				131	6	3
Amount of Funds, 31st March 1872, viz.:—						
Mortality Experience Fund	329	8	3			
Messenger Legacy ,,	245	5	1			
Hardy Memorial ,,	212	7	11			
Brown Prize ,,	214	14	10			
General ,,	1151	13	1			
				2153	9	2
				£2,812	11	1

Balance Sheet, 31*st March* 1872.

	£	s.	d.
…ance … Fu…	329	8	3
	245	5	
	212	7	11
	214	14	10
	1151	13	1

	£	s.	d.	£	s.	d.
Union Bank Deposit Account				600	0	0
Three per-cent Consols (£1100)				1008	19	3
East India 10½ per-cent Stock (£95. 15*s.* 5*d.*)				200	0	0
Cash, viz.:—						
London and Westminster Bank	318	19	0			
Petty Cash	8	14	11			
				327	13	11
Arrears of Subscriptions				16	16	0
				£2,153	9	2

29*th April* 1872.

Examined and found correct,

(Signed) { STEPHEN H. EMMENS, J. CLIFFORD HOPKINSON, } *Auditors.*

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

On the Means of dispensing with Extra Premiums for Deteriorated Health. By W. M. Makeham, *Fellow of the Institute of Actuaries.*

THERE are few things so discouraging to an agent, after successfully canvassing for an assurance, as to find that the medical examiner is unable to recommend the case (which has, perhaps, been secured by a serious expenditure of time and trouble), without an addition to the ordinary rate of premium. Knowing the difficulty he has experienced in persuading the proposer to take the decisive step, the agent feels that, in nine cases out of ten, the demand for an extra premium is tantamount to the rejection of the proposal—for, somewhat unreasonably, perhaps, instead of impressing more strongly upon the party concerned the importance of life assurance, the effect is generally to irritate and annoy—or even to create in his mind a suspicion that he is treated unfairly in being required to pay a higher rate than other persons of the same age.

The recently published experience of Assurance Offices shows, beyond dispute, that lives which have been taken at an extra premium are, upon the whole, materially worse than those taken at the ordinary rates—although, perhaps, not to the extent which might have been supposed. Comparing the average duration of life among the healthy and deteriorated lives respectively, I find

that, roughly speaking, the results may be thus stated:—At the ages

15 to 25—The difference corresponds to an addition of 6 years to the age.					
25 to 35	,,	,,	,,	5	,,
35 to 45	,,	,,	,,	4	,,
45 to 55	,,	,,	,,	3	,,
55 to 65	,,	,,	,,	2	,,
65 to 75	,,	,,	,,	1	,,

while from and after the age of 80 the two tables about coincide—indeed, in some instances, the deteriorated lives have rather the advantage—an anomaly due probably to the paucity of the lives under observation at these advanced ages.

It is to be regretted that the average amount of extra premium charged in the cases from which these results were deduced has not been ascertained. Nevertheless, I think it is evident that, while the existence of a higher mortality is indisputable, it is covered, and probably considerably more than covered, by the extra charges usually made. The conclusion, therefore, to be drawn from the experience in question, seems to be that while, perhaps, the imposition of extra premiums may be somewhat more sparingly resorted to than formerly, still it would be highly injudicious to dispense with them in cases where the existence of decidedly unfavourable circumstances is evident. For although, if a fair general average were obtained, the tabular rates would be fully adequate to cover the normal proportion of such lives, yet an office known to adopt a practice of this kind would doubtless be resorted to by an inferior class of lives, in numbers sufficient to disturb the average, and thus its aggregate mortality might by this means be affected to a serious extent.

When a medical examiner is called upon to form an estimate of the *amount* of deterioration in the value of a life proposed for assurance, he generally expresses it in the number of years which, in his opinion, should be added to the real age in fixing the rate of premium—thereby implying (whether intentionally or not) that the nature of the deterioration is such, that while the *immediate* risk is somewhat increased, yet that the principal reason for the addition lies in the probability that the *future* risk will be increased to a much greater extent as the life gets older—by the advent, as it were, of a premature old age.

This, consequently, is the way in which deterioration has hitherto been exclusively treated by Assurance Offices. The life is

assumed to be of the age for which the rate of premium is assessed, and in any subsequent calculations the age corresponding thereto is taken as the *actual* age of the life assured.

When this theory of the nature of the deterioration coincides with the fact—that is to say, when the aggregate extra risk is really distributed in a progression increasing with the age—there is, perhaps, no more satisfactory way of dispensing with the necessity of imposing an extra premium than that of substituting an Endowment Assurance (that is, an assurance payable at death, or upon the attainment of a certain stated age) in lieu of an ordinary assurance payable at death only.

For instance, the net single premium required to provide the sum of £100 at the death of a person aged 30 is (by the Carlisle 4 per-cent table) £31. 6*s*. 9*d*., while in the case of a person aged 40 it is £38. 3*s*. 7*d*. Hence it appears that in an ordinary assurance effected by a single premium, an addition of 21·8, or nearly 22 per-cent, is necessary to cover the increased risk equivalent to an addition of 10 years to the age.

Again, the single premium (age 30) required to provide £100 to be paid on the attainment of the age of 50—that is to say, at the expiration of the term of 20 years,—or at death, if that event shall happen previously, is £50. 6*s*. 3*d*.; while for age 40 a similar assurance (for the same *term of years*) requires a single premium of £51. 12*s*. 3*d*. In this case, therefore, an addition of about 2½ per-cent (2·6) is sufficient to cover the increased risk equivalent to precisely the same addition (viz. 10 years) to the age.

It is evident, therefore, that if the effects of deterioration are truly represented by an addition of a certain number of years to the age, a damaged life, which could not safely be taken by the ordinary table without a considerable addition to the usual rate, might without any very material increase of liability be accepted for an endowment assurance at the ordinary rate of premium.

The reason for this somewhat paradoxical result will be found to be as follows. An endowment assurance consists of two entirely distinct parts, viz., first, an ordinary life assurance for the given term, and secondly, a sum payable only in the event of the proposer *surviving* a certain age; and the premium charged really consists of the sum of the premiums required for these separate contingencies. Now, by the assumption made as to the distribution of the extra risk, the portion of the premium required to cover the term assurance, although increased to some extent, is by no means so much increased as it would be under an assurance for the whole

L 2

term of life. But the other portion of the premium, namely, that required to provide for the sum assured in the event of the proposer *surviving* the given age, is evidently *diminished* by the assumed increase of mortality; and thus it happens that although the two contrary operating causes do not quite destroy each other, they suffice in many cases to reduce the balance to an amount so small that it may with safety be neglected.

In a letter which I addressed in January 1868, to the editor of the *Journal*, and which appeared in the 14th volume, I had occasion to point out that the hypothesis of deterioration involved in the assumption of a given addition to the age, was by no means necessarily adapted to all cases, and that some reform in the very primitive methods usually adopted in dealing with extra risks was needed. The subject, I am glad to see, has recently been taken up, at the point where I left it, by Mr. Macfadyen, who, in a paper read before the Institute early in the present year, also urged the necessity of a more scientific mode of dealing with these cases. Having been the first to broach the subject, it was gratifying to me to find that Mr. Macfadyen's paper was favourably received by the leading members of the Institute, a circumstance which indicates that the question is now ripe for a further advance in the path of improvement. As Mr. Macfadyen's mode of treating the subject differs in some respects from mine, I propose here to explain the views which further consideration has led me to take of the matter.

As I have already explained, the hypothesis of an addition to the age involves the assumption that the extra risk of mortality increases year by year as the life gets older; or, in other words, that the aggregate extra risk is distributed in an *increasing* progression. Now, it is extremely probable that such may be the case with many forms of deterioration; to which, therefore, the hypothesis in question would be perfectly applicable. But it is equally probable, I think, that in other cases the additional risk may be supposed to distribute itself more *uniformly* over the entire period of life, or indeed may even form a *decreasing* progression; that is to say, we may easily suppose a case where the unfavourable feature is such as to constitute a considerable addition to the *present* risk, but which, if the person should survive a certain number of years, would disappear, or at least become of much less importance. Cases of these two last-mentioned descriptions are evidently not so well met by the endowment assurance plan as the former, and it would seem at first sight that in dealing with them we have no

option but to insist upon the payment of an extra premium corresponding to the extent of the deterioration.

The following little table, extracted from my letter of January 1868, before referred to, affords a good illustration of the view just explained. It shows the amount of extra premium for different descriptions of assurance required upon the assumption of a constant or uniformly distributed extra rate of mortality of about 2 per-cent per annum, with a profit loading of 30 per-cent.

Age.	ANNUAL EXTRA PREMIUMS PER-CENT.			
	Assurances for the Term of One Year.	Whole Life Assurances. Premiums payable during		Endowment Assurances (Age of 60).
		Life.	Ten Years.	
	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	2 6 9	2 1 0	3 6 2	1 17 9
30	2 6 8	1 19 10	2 19 3	1 15 1
40	2 6 6	1 18 3	2 11 6	1 11 6
50	2 6 7	1 16 1	2 2 7	1 5 9
60	2 5 6	1 15 8	1 16 0	

From these examples it will be seen that, quite contrary to the former case, the extra premium is greater for a short term assurance than for one effected for the whole term of life; while for an endowment assurance it is nearly as great. If results so different are produced by changing an *increasing* into a *uniform* distribution of extra risk, it is evident that the substitution of a *decreasing* distribution would present a contrast still more striking.

In the discussion, however, which followed the reading of Mr. Macfadyen's paper, I find a plan described by two or three members, which seems to me likely to afford a satisfactory substitute for the endowment assurance plan, and which, indeed, becomes more advantageous to the Society precisely in those cases where the latter plan fails to afford the requisite compensation. The only question is whether the method suggested is likely to meet with the acceptance of the public, a point upon which the testimony of the gentlemen who described the process (and who spoke from experience of its actual working) is entirely favourable. I am further induced to look hopefully upon the practicability of the scheme, by the fact that some very energetic and successful assurance agents have specially urged it upon my attention, as one likely to remove a serious difficulty experienced by them in canvassing for business.

The plan in question consists in making a certain deduction

from the amount assured for each year that the actual duration of the life falls short of a given period of time. Thus, supposing 20 years to be the probationary period fixed upon, the sum assured will be paid in full if death should take place *after* the expiration of that period, but if death should take place during the twentieth year, a small sum is deducted from the amount assured; if in the nineteenth year, twice that sum is deducted; if in the eighteenth year, three times, and so on. These deductions are supposed to be sufficient to compensate the Office for the non-payment of the extra premiums required.

It will thus be seen that the scheme really consists in granting an *increasing* insurance, at the rate of premium required for a *uniform* assurance equal to the maximum amount; and if the rate of increase be so regulated that the premium required to secure the increasing assurance, calculated for the *increased* age of the life, shall be equal to the premium for the uniform assurance calculated for the *actual* age, the Society gets the full benefit of the extra premium, without any actual additional payment by the policyholder.

For the purpose of investigating the practical working of the scheme, I have calculated the following rates of premium required to secure an increasing assurance, commencing at different sums, and progressing by equal annual increments during 25 years, when it reaches a maximum of £100. The table used in the calculation is the Carlisle 4 per-cent.

Age.	Proportion which the first year's Assurance bears to the maximum of £100.			
	One half.	Three sevenths.	Three eighths.	One third.
	Annual Premium.	Annual Premium.	Annual Premium.	Annual Premium.
20	1·1286	1·1014	1·0810	1·0652
35	1·6968	1·6503	1·6155	1·5884
50	2·6940	2·5983	2·5265	2·4707

Now, comparing these premiums with those required for a uniform assurance of £100, I find that the three terms of the first column correspond to the ordinary premiums for the ages 14, 29, and 44, respectively; the three terms of the second column to the ages 13, 28, and 43; the three terms of the third column to the ages 12, 27, and 42; and, finally, the three terms of the fourth column to the ages 11, 26, and 41. Of course it is not intended to imply that the correspondence in any case is *exact;* but the difference is always within a fraction of a year. From this remark-

able coincidence, I deduce the very simple and convenient practical *rule:*

When the proportion which the first year's assurance bears to the maximum is—

<table>
<tr><td>As Three to Six</td><td rowspan="4">the diminution of the amount at risk, during the first twenty-five years of the assurance, is equivalent to an addition of</td><td>Six years</td><td rowspan="4">to the age.</td></tr>
<tr><td>Three to Seven</td><td>Seven years</td></tr>
<tr><td>Three to Eight</td><td>Eight years</td></tr>
<tr><td>Three to Nine</td><td>Nine years</td></tr>
</table>

We have thus the means of determining very readily the rate of deduction corresponding to any given addition to the age. Thus, suppose that the medical officer recommends an addition, say, of 10 years to the actual age,—the first year's assurance will be to the maximum as three to ten—that is to say, £30 for every £100 assured; and the difference (£70) divided by 25, namely, £2. 16*s*., is the annual rate of deduction for every year which the actual duration of the life falls short of 25 years.

The term of 25 years may, perhaps, be considered too long for ages over 44, as it would then exceed the average duration or expectation of life. I have therefore deduced a similar rule for the terms of 20 and 15 years respectively. The general rule thus extended is as follows:—add 5 years to the probationary term and divide the result by 10. The ratio which the quotient bears to the number of years added to the age, is the ratio which the first year's assurance bears to the maximum. Thus, suppose 5 years to be added to the age, and the probationary term to be 15 years. Twenty divided by 10, or 2, and 5 express the required proportions —that is to say, the first year's assurance should be $\frac{2}{5}$ths of the maximum amount, or £40 for every £100 assured. The difference, £60, divided by 15, gives £4 for the yearly deduction in this case.

In general it will be found convenient to adopt the following scale in fixing the probationary term:—

25 years for ages under 40.
20 years for ages between 30 and 50.
15 years for ages between 40 and 60.

As, however, the rule fails when the number of years added is small and the probationary term long, the latter should never exceed *five* times the integer next less than the former. That is to say, the first year's assurance should never be more than *one-half* of the maximum.

To conclude, it will be observed that these equivalents are deduced upon the supposition that the deterioration is truly represented by an addition of a given number of years to the age—that

is to say, upon the supposition of an extra risk comparatively small at first, but increasing with the age of the life assured. It is evident, therefore, that if the aggregate extra risk on the other hand be distributed equally at all ages—and still more, if it be supposed to be a *decreasing* extra risk—then the equivalents deduced by the preceding rules will be in favour of the office—for it is in the early years that the deduction from the sum at risk is greatest. Now, it will be remembered that precisely the reverse of this is the case with the endowment assurance plan, which we have seen is strictly applicable only to the hypothesis of an increasing extra risk. The new method, which may be designated the probationary plan, has therefore the advantage of being universally applicable, and may safely be used in cases where the endowment assurance plan would not adequately meet the extra risk.

The law of the *distribution of risk* (*i.e.* of the force of mortality) is an important element in problems depending upon the *average duration* of life. The probability of living over any given term is, however, in no way affected by it,—the latter being a function only of the *aggregate sum of the force of mortality prevailing during the given term.* Thus, if K_x denote the aggregate force of mortality from age 0 to x,—that is to say, if $K_x=\int_0^x\mu_x dx$,—we shall have ${}_xp_0=\epsilon^{-K_x}$, where ϵ is the base of the Neperian system of logarithms; for $-\frac{1}{l_x}\cdot\frac{dl_x}{dx}=\mu_x \quad \therefore \quad -\log l_x=\int\mu_x dx$, and $\int_0^x\mu_x dx= -\log l_x+\log l_0=-\log\frac{l_x}{l_0} \quad \therefore K_x=-\log {}_xp_0$, the logarithms being Neperian.

Postscript.—In Mr. Macfadyen's paper, above referred to, it is stated that in my letter of January 1868, I have taken it for granted that a person having been subjected to Extra Risk (from residence in a foreign climate) will be in exactly the same state of health, after return to Europe, as if the extra risk had not been incurred. With reference to this remark, I have to observe, that in the letter in question I have not advanced, either by implication or otherwise, any opinion whatever on the subject. The object of my investigation was to expose the error of the present system, not on abstract grounds, but upon the bases of its own assumptions. The question raised by Mr. Macfadyen is no doubt an important one, but as it admits of a very obvious and simple mode of treatment, upon the principles I have advocated, and as it in no way affected the truth of my conclusions, I should certainly not have thought it necessary to complicate my subject by adverting to it.

*** The above paper will no doubt prove practically useful to the managers of the companies which adopt the method of charging for impaired health therein discussed; but it appears to us to be even more valuable to the student of the theory of life contingencies, as suggesting the course which future investigations into the mortality among under-average lives should follow. Mr. Makeham's investigations clearly demonstrate that the point to be ascertained is whether the increased mortality among such lives is more apparent immediately after the grant of the policy or in later years; and any future investigation into such mortality must be considered incomplete that does not give special attention to this point.—Ed. *J. I. A.*

On the Construction of a Graduated Table of Mortality from a Limited Experience. By S. C. Chandler, Jr., *Actuary of the Continental Life Insurance Company, of New York.*

[Reprinted from the *Spectator* of New York and Chicago.]

THE discussion of the methods of dealing with the mortuary statistics of a life insurance company has been hitherto almost entirely neglected by American actuarial writers. This is not a little singular, considering the attention bestowed on other subjects of not greater difficulty or importance. Viewing the large and rapidly-growing body of data already accumulated by American companies, the problem is not of less concern to us than to our English and German brethren, to whom, notwithstanding, we are indebted for the principal contributions to this branch of actuarial science.

In bringing to notice, with great diffidence, the methods which have suggested themselves to me as the most practicable and efficient for the treatment of a limited mortuary experience, I venture to express the hope that the way may be opened for a general discussion of the subject.

There is at present a very great diversity of opinion among actuaries in regard to what is commonly termed the "graduation" of the results of observations of human mortality. Many object to any interference with the naked results of observations. Prominent among these may be mentioned the late Professor de Morgan, and, if I mistake not, Mr. David Chisholm. The language of the former on this point,* though decided, had greater force at the

* Treatise on Probabilities, p. 162.

date it was written than it has now, and I am not sure that his views were not subsequently modified.

In a second class may be placed those actuaries who, while admitting the desirableness of removing the grosser deviations of the observations at individual ages, due to the limited data employed, prefer to effect that object by processes which are independent of any hypothetical assumption as to the nature of the law of mortality.

Finally, there is a third class who deem the approximate knowledge we possess of the nature of the law of mortality sufficient to justify the use of processes based, to a greater or less degree, upon an assumed relation between the mortality at different ages, for the purpose of securing the required adjustment.

It is not intended here to enter upon an examination of the subject in its general relations. I will only say, with reference to the object directly in view, that when the data are very limited, the raw results of observation are unfit for use in many of the comparisons and deductions which it is desirable to make; while the methods which exclude considerations of the character of the law of mortality, of necessity more or less desultory in their nature, become too uncertain in their application to afford trustworthy results.

If there be a general law governing human mortality, and if the variations from this law that obtain among any given body of individuals, to whatever circumstances due, are such as do not affect the form of the mathematical expression of the general law, but merely the values of the constant elements of that expression, —it is not essential for the purpose of eliminating the errors in a limited body of observations, that the true form of the general expression should be known. Any expression, however arbitrary, will be of service, if we possess the means of estimating the accuracy with which it represents the observations, and of determining the limits within which the deviations from the true law, being inappreciable, may be neglected.

To apply this principle to the case of a life insurance company, we observe that the ages at which lives are subject to observation are ordinarily comprised within a period of forty-five or fifty years. In most of our American companies the limits of age for the acceptance of risks lie between 15 and 25, and between 60 and 70. The experience at ages above the latter limit, pertaining to risks which have continued in force from the younger ages, is, even in the older companies, exceedingly scanty. So that, even if we were in possession of the true formula, we might not have sufficient

material for fixing the values of the constants required for its numerical application outside the limits named.

Now, in the assumption that the force of mortality is the resultant of two single forces, one constant at all ages, the other increasing in geometrical progression with the age, we have a relation which, though it requires considerable modification to represent the mortality in infancy and childhood, and in extreme old age, conforms very closely to the observations on which our best life tables are founded, within the limits of age above stated. It seems, therefore, to be admirably adapted for the present purpose, being at once simple, and so far as the application here to be made of it is concerned, as accurate in its results as the true law of which it is but the approximate expression.

According to the hypothetical principle just enunciated, the force of mortality at the age x may be represented by an equation of the form

$$F_x = a + bq^x \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The credit of this formula is due to Mr. Makeham, to whose labours in this department the actuarial world owes so much. In the *Journal of the Institute of Actuaries* for October 1871, Mr. Makeham has given a mode of employing it, with a numerical application to a case where the number of deaths is but 27, and the years of life exposed, for all ages, less than 3000. The method of Mr. Makeham, though exceedingly simple, is, however, open to the grave objection that the values of the constants a and b depend entirely upon the age t, which is chosen arbitrarily. There would thus be as many values for the mortality at any age as there are possible combinations of the equations of condition into two groups. Moreover, it furnishes no criterion of the accuracy of the determination. In requiring that the sum of the deaths in each of the arbitrarily-chosen groups shall be the same for the adjusted as for the unadjusted numbers, Mr. Makeham assumes, in effect, two rigorous equations of condition, which, when q is known, of course define a and b.

The best mode of procedure, it seems to me, even where the data are as scanty as in the case exemplified by Mr. Makeham, is to form equations of condition assigning appropriate weights at each age, and then deduce from the whole series the most probable values of the constants. I propose in this paper to give the necessary formulas for accomplishing this, with such explanation and illustration as shall enable computers not familiar with the calculus of errors to make a practical application of them.

Representing the true annual rate of mortality at the age x by m_x, we shall have from (1), since $m_x = F_{x+\frac{1}{2}}$, very nearly,

$$a + bq^{x+\frac{1}{2}} = m_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If in any series of observations we denote the years of life exposed and the actual number of deaths at age x by E_x and d'_x, respectively, and the corresponding ratio $\frac{d'_x}{E_x}$ by m'_x, we shall have for every age under observation an approximate equation of condition of the form

$$a + bq^{x+\frac{1}{2}} - m'_x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which a, b, and q are to be found.

In most cases that arise in practice the experience will be insufficient to determine the constant q with any accuracy. But the value of this element is nearly the same in all tables of mortality. Its logarithm may be taken $= \cdot 04$ without appreciable error. This reduces the number of unknown quantities to two.

The weight of the observed probability of dying in a year found from E_x observations is proportional to

$$w_x = \frac{E_x}{m_x(1 - m_x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Multiplying the equations of the form of (3) each by the square root of its weight, we have

$$\left.\begin{array}{l} \sqrt{w_x}\,a + \sqrt{w_x}\,q^{x+\frac{1}{2}}b - \sqrt{w_x}\,m'_x = 0 \\ \sqrt{w_{x+1}}\,a + \sqrt{w_{x+1}}\,q^{x+\frac{3}{2}}b - \sqrt{w_{x+1}}\,m'_{x+1} = 0 \\ \sqrt{w_{x+2}}\,a + \sqrt{w_{x+2}}\,q^{x+\frac{5}{2}}b - \sqrt{w_{x+2}}\,m'_{x+2} = 0 \\ \quad \&c., \qquad \&c., \qquad \&c., \end{array}\right\} \quad . \quad . \quad (5)$$

all of equal weight, and from these the normal equations

$$\left.\begin{array}{l} \Sigma w_x a + \Sigma w_x q^{x+\frac{1}{2}} b - \Sigma w_x m'_x = 0, \\ \Sigma w_x q^{x+\frac{1}{2}} a + \Sigma w_x q^{2x+1} b - \Sigma w_x q^{x+\frac{1}{2}} m'_x = 0 \end{array}\right\} \quad . \quad . \quad (6)$$

from which the most probable values of a and b may be obtained by elimination.

If it is desired to determine q, we may proceed as follows. Calling m''_x the hypothetical value of the rate of mortality which results from the substitution of the approximate values a', b', q', in (2), and substracting the resulting equation

$$a' + b'q'^{x+\frac{1}{2}} = m''_x$$

from (2), we obtain

$$a - a' + bq^{x+\frac{1}{2}} - b'q'^{x+\frac{1}{2}} = m'_x - m''_x.$$

Putting
$$\left.\begin{array}{l} a = a' + X, \; q = q' + Z \\ b = b' + Y, \; m'_x = m''_x + \mu_x \end{array}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

and neglecting the terms involving the squares and higher powers of Z in the development, we find

$$X+q'^{x+\frac{1}{2}}Y+b'(x+\tfrac{1}{2})q'^{x-\frac{1}{2}}Z-\mu_x=0 \quad . \quad . \quad (8)$$

Reducing to the unit of weight, as before,

$$\left.\begin{array}{l}\sqrt{w_x}X+\sqrt{w_x}q'^{x+\frac{1}{2}}Y+\sqrt{w_x}b'(x+\tfrac{1}{2})q'^{x-\frac{1}{2}}Z-\sqrt{w_x}\mu_x=0 \\ \qquad \&c., \qquad\qquad \&c., \qquad\qquad \&c.,\end{array}\right\} \quad . \quad . \quad . \quad (9)$$

whence we derive the normal equations for determining the corrections to the assumed values of the constants,

$$\left.\begin{array}{l}\Sigma w_x X+\Sigma w_x q'^{x+\frac{1}{2}}Y+\Sigma w_x b'(x+\tfrac{1}{2})q'^{x-\frac{1}{2}}Z-\Sigma w_x\mu_x=0 \\ \Sigma w_x q'^{x+\frac{1}{2}}X+\Sigma w_x q'^{2x+1}Y+\Sigma w_x b'(x+\tfrac{1}{2})q'^{2x}Z-\Sigma w_x q'^{x+\frac{1}{2}}\mu_x=0 \\ \Sigma w_x b'(x+\tfrac{1}{2})q'^{x-\frac{1}{2}}X+\Sigma w_x b'(x+\tfrac{1}{2})q'^{2x}Y+\Sigma w_x b'^2(x+\tfrac{1}{2})^2 q'^{2x-1}Z \\ \qquad\qquad\qquad\qquad -\Sigma w_x b'(x+\tfrac{1}{2})q^{x-\frac{1}{2}}\mu_x=0\end{array}\right\} \quad (10)$$

and from these the most probable values of a, b, and q, by (7). Unless Z is very small, it may be necessary to make another approximation, using the results of this solution for the assumed values.

For the calculation of the numerical values of the weights, we may employ provisional values of a and b found by taking the definite sums of the equations of the form given in (3), from the youngest age n to an arbitrarily-chosen age t, and from t to the oldest age ω, or

$$\left.\begin{array}{l}\nu a+b\Sigma_n^{t-1}q^{x+\frac{1}{2}}=\Sigma_n^{t-1}m'_x \\ \nu' a+b\Sigma_t^{\omega}q^{x+\frac{1}{2}}=\Sigma_t^{\omega}m'_x\end{array}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

where ν and ν' are the numbers of equations in each group. Then, from (2) and (4) the values of w can be calculated. If the results deduced from (6) differ much from the approximate values employed in finding the weights, the latter may be computed anew and applied as before.

To exemplify the method just developed, I propose to apply it to the data* in columns (2) and (3) of the accompanying table. Column (4) contains the actual ratios of mortality deduced from these.

Our first business will be to compute the weights. Taking $\log q=\cdot04$ and $t=30$, we find

$$\nu=16, \quad \Sigma_{14}^{29}q^{x+\frac{1}{2}}=\ 132\cdot61, \quad \Sigma_{14}^{29}m'^{x}=\cdot1283,$$
$$\nu'=41, \quad \Sigma_{30}^{70}q^{x+\frac{1}{2}}=7336\cdot79, \quad \Sigma_{30}^{70}m'^{x}=\cdot8381;$$

which, substituted in (11), give

$$a=\cdot00732, \quad b=\cdot0000728.$$

* The published experience of the [New York] Mutual Life for the first fifteen years.

Construction of a Graduated Table of Mortality from a Limited Experience.

Illustration.

(1) x	(2) E_x	(3) d'_x	(4) m'_x	(5) $q^{x+\frac{1}{2}}$ ($\log q = \cdot 04$)	(6) $\frac{1}{m_x(1-m_x)}$	(7) w_x	(8) m_x	(9) $m_x - m'_x$	(10) m_x	(11) l_x	(12) d_x	(13) x
10							·0076029			1000000	7603	10
11							·0076194			992397	7561	11
12							·0076375			984836	7522	12
13							·0076574			977314	7484	13
14	22·10		·0000	3·802	134	2961	·0076679	+ ·00768		969830	7447	14
15	38·52		·0000	4·169	133	5123	·0077030	+ ·00770		962383	7413	15
16	42·01		·0000	4·571	132	5545	·0077291	+ ·00773		954970	7381	16
17	54·58		·0000	5·012	132	7204	·0077578	+ ·00776		947589	7351	17
18	61·60		·0000	5·495	131	8070	·0077892	+ ·00779		940238	7324	18
19	109·45	4	·0365	6·026	131	14338	·0078237	− ·02868		932914	7299	19
20	172·90	1	·0058	6·607	130	22477	·0078615	+ ·00206		925615	7277	20
21	271·68	4	·0147	7·244	129	35047	·0079030	− ·00680		918338	7257	21
22	396·32	5	·0126	7·943	128	50730	·0079484	− ·00465		911081	7242	22
23	596·83	4	·0067	8·710	127	75797	·0079982	+ ·00130		903839	7229	23
24	806·13	10	·0124	9·550	126	101573	·0080529	− ·00435		896610	7220	24
25	1046·36	11	·0105	10·471	125	130795	·0081128	− ·00239		889390	7215	25
26	1290·30	11	·0085	11·482	124	159996	·0081785	− ·00032		882175	7215	26
27	1589·81	8	·0050	12·589	123	195550	·0082505	+ ·00325		874960	7219	27
28	1877·73	11	·0059	13·804	122	229080	·0083295	+ ·00243		867741	7228	28
29	2069·02	20	·0097	15·136	120	248280	·0084161	− ·00128		860513	7242	29
30	2312·85	26	·0112	16·596	118	272910	·0085110	− ·00269		853271	7262	30
31	2503·43	21	·0084	18·197	117	292910	·0086152	+ ·00022		846009	7288	31
32	2700·65	31	·0115	19·953	115	310580	·0087293	− ·00279		838721	7322	32
33	2811·82	26	·0092	21·878	114	320550	·0088545	− ·00035		831399	7362	33
34	2994·62	35	·0117	23·988	112	335400	·0089917	− ·00271		824037	7409	34
35	3020·93	29	·0096	26·303	110	332300	·0091422	− ·00046		816628	7466	35
36	3049·80	35	·0115	28·840	107	326320	·0093072	− ·00219		809162	7531	36
37	3067·96	23	·0075	31·623	105	322140	·0094882	+ ·00199		801631	7606	37
38	3049·61	30	·0098	34·674	102	311090	·0096865	− ·00011		794025	7691	38
39	2967·64	31	·0104	38·019	100	296760	·0099041	− ·00050		786334	7788	39
40	2842·58	13	·0046	41·687	98	278570	·0101426	+ ·00554		778546	7897	40
41	2686·54	21	·0078	45·709	95	255220	·0104041	+ ·00260		770649	8018	41
42	2551·34	21	·0082	50·119	92	234730	·0106909	+ ·00249		762631	8153	42
43	2407·72	28	·0116	54·954	89	214290	·0110053	− ·00059		754478	8303	43
44	2254·94	23	·0102	60·256	86	193920	·0113500	+ ·00115		746175	8469	44
45	2077·77	21	·0101	66·069	83	172450	·0117280	+ ·00163		737706	8652	45
46	1899·11	15	·0079	72·444	80	151930	·0121425	+ ·00424		729054	8853	46
47	1695·42	20	·0118	79·433	77	130550	·0125970	+ ·00080		720201	9072	47
48	1534·69	23	·0150	87·096	74	113570	·0130953	− ·00190		711129	9312	48
49	1404·73	18	·0128	95·499	71	99736	·0136417	+ ·00084		701817	9574	49
50	1243·14	15	·0121	104·713	68	84534	·0142408	+ ·00214		692243	9858	50
51	1090·47	17	·0156	114·815	65	70880	·0148977	− ·00070		682385	10166	51
52	949·08	12	·0126	125·893	62	58842	·0156180	+ ·00302		672219	10499	52
53	822·56	15	·0182	138·038	58	47775	·0164078	− ·00179		661720	10857	53
54	726·99	16	·0220	151·356	55	39984	·0172738	− ·00473		650863	11243	54
55	621·81	12	·0193	165·959	52	32334	·0182233	− ·00108		639620	11656	55
56	543·42	13	·0239	181·970	49	26628	·0192644	− ·00464		627964	12098	56
57	448·37	9	·0201	199·526	46	20625	·0204060	+ ·00031		615866	12567	57
58	370·62	6	·0162	218·776	43	15937	·0216578	+ ·00546		603299	13066	58
59	311·74	3	·0096	239·883	40	12470	·0230302	+ ·01343		590233	13593	59
60	259·19	7	·0270	263·027	38	9849	·0245351	− ·00246	·0245351	576640	14148	60
61	212·50	12	·0565	288·403	36	9463	·0261852	− ·03031	·0258840	562492	14560	61
62	167·33	5	·0298	316·228	33	5522	·0279945	− ·00181	·0274620	547932	15047	62

Construction of a Graduated Table of Mortality, &c.—(continued).

(1) x	(2) E_x	(3) d'_x	(4) m'_x	(5) $q^{x+\frac{1}{2}}$ ($\log q = \cdot04$)	(6) $\frac{1}{m_x(1-m_x)}$	(7) w_x	(8) m_x	(9) $m_x - m'_x$	(10) m_x	(11) l_x	(12) d_x	(13) x
63	131·62	9	·0684	346·737	31	4080	·0299784	−·03842	·0293738	532885	15653	63
64	104·84	5	·0477	380·189	29	3040	·0321536	−·01555	·0317080	517232	16400	64
65	86·16	5	·0580	416·869	27	2326	·0345387	−·02346	·0345387	500832	17298	65
66	67·19	3	·0446	457·088	25	1680	·0371539	−·00745	·0379247	483534	18338	66
67	49·66	1	·0201	501·187	23	1142	·0400214	+·01992	·0419092	465196	19496	67
68	35·00	1	·0286	549·541	21	735	·0431656	+·01457	·0465209	445700	20735	68
69	26·98	1	·0371	602·560	20	540	·0466132	+·00951	·0517722	424965	22001	69
70	20·07	1	·0498	660·694	18	361	·0503938	+·00059	·0576631	402964	23236	70
71	12·56		·0000	724·436	17	214	·0545381	+·05454	·0641746	379728	24369	71
72	11·55	1	·0866	794·328	16	185	·0590829	−·02752	·0712751	355359	25328	72
73	6·71		·0000	870·963	15	101	·0640661	+·06407	·0789170	330031	26045	73
74	6·95	1	·1439	954·993	14	97	·0695800	−·07437	·0870381	303986	26459	74
75	4·09		·0000	1047·130	13	53	·0755201	+·07552	·0955602	277527	26520	75
76	3·96		·0000	1148·150	12	48	·0820902	+·08209	·1043906	251007	26203	76
77	2·71		·0000	1258·930	10	27	·0892932	+·08929	·1134212	224804	25498	77
78	1·03	1	·9709	1380·380	10	10	·0971910	−·87371	·1225288	199306	24420	78
79									·1315750	174886	23011	79
80									·1404063	151875	21324	80
81									·151436	130551	19770	81
82									·163194	110781	18079	82
83									·175913	92702	16307	83
84									·189678	76395	14491	84
85									·205095	61904	12696	85
86									·222480	49208	10948	86
87									·242234	38260	9268	87
88									·265274	28992	7691	88
89									·292382	21301	6228	89
90									·323730	15073	4879	90
91									·360987	10194	3680	91
92									·405263	6514	2640	92
93									·457227	3874	1771	93
94									·516306	2103	1086	94
95									·584270	1017	594	95
96									·648649	423	274	96
97									·692308	149	105	97
98									·750000	46	35	98
99									1·000000	11	11	99

Similarly, when $t=46$,

$$a = \cdot00713,\quad b = \cdot0000749.$$

For determining the weights we shall adopt

$$a = \cdot00726,\quad b = \cdot000074;$$

from which we compute the factor $\frac{1}{m_x(1-m_x)}$ in column (6) to be multiplied into E_x to obtain the weight w_x in column (7). We can now form the normal equations (6). For the purpose of this illustration, five places of logarithms have been deemed sufficient

in calculating the auxiliaries $w_x q^{x+\frac{1}{2}}$, $w_x m'_x$, &c. Making the necessary summations we get

$$\Sigma w_x = 6,707,974, \quad \Sigma w_x q^{x+\frac{1}{2}} = 278,341,720, \quad \Sigma w_x m'_x = 67,952,$$
$$\Sigma w_x q^{2x+1} = 23,249,674,696, \quad \Sigma w_x q^{x+\frac{1}{2}} m'_x = 3,580,417.$$

Introducing these in (6) and solving, we get the most probable values,—

$$a = \cdot 0074319, \quad b = \cdot 000065025.$$

Column (8) shows the rates of mortality calculated from these by means of (2), and column (9) the differences (computed *minus* observed) from the actual ratios.

To ascertain the probable errors of the determination we have (n being the number of equations of condition) the mean error of an observation of the weight unity,—

$$\epsilon = \sqrt{\frac{\Sigma w_x (m_x - m'_x)^2}{n-2}} = \sqrt{\frac{80 \cdot 495}{63}} = \pm 1 \cdot 1304:$$

also the weights of the determinations of a and b,

$$w_a = \Sigma w_x - \frac{(\Sigma w_x q^{x+\frac{1}{2}})^2}{\Sigma w_x q^{2x+1}} = 3,375,708,$$

$$w_b = \Sigma w_x q^{2x+1} - \frac{(\Sigma w_x q^{x+\frac{1}{2}})^2}{\Sigma w_x} = 11,700,130,696.$$

From these the mean errors of a and b are found to be

$$\epsilon_a = \frac{\epsilon}{\sqrt{w_a}} = \pm \cdot 0006152,$$

$$\epsilon_b = \frac{\epsilon}{\sqrt{w_b}} = \pm \cdot 000010450;$$

and, from the constant relation between the mean and probable errors, $r = \cdot 6745\,\epsilon$,

$$r_a = \pm \cdot 0004150, \quad r_b = \pm \cdot 00007049.$$

Recapitulating, then, the results of this determination,

$a = \cdot 0074319$ with the probable error of $\pm \cdot 0004150$.
$b = \cdot 000065025$,, ,, ,, $\pm \cdot 000007049$.

To illustrate the use of the formulas for finding the most probable value of q at the same time with those of a and b, let us assume the approximate values—

$$a' = \cdot 00726, \qquad b' = \cdot 000074,$$
$$q' = 1 \cdot 096,478 \text{ or } \log q' = \cdot 04,$$

With the value m'' of the mortality rates corresponding to these assumptions we find the values of μ from (7). Then, performing

the requisite multiplications and summations, we deduce the following normal equations, according to formulas (10),

$$
\begin{aligned}
6{,}707{,}974X + 278{,}341{,}720Y + 826{,}858\ Z - 1{,}267\ &= 0,\\
278{,}341{,}720X + 23{,}249{,}674{,}696Y + 81{,}926{,}500\ Z - 159{,}723\ &= 0,\\
826{,}858X + 81{,}926{,}500Y + 298{,}367{\cdot}6Z - 460{\cdot}6 &= 0,
\end{aligned}
$$

whence

$$X = +{\cdot}0001829,\quad Y = -{\cdot}000005229,\quad Z = -{\cdot}0012554,$$

are the corrections to be added to a', b', and q'.

The correction Z comes out with the large mean error $\pm{\cdot}001070$. Also, the mean error of an observation of the weight unity is

$$\epsilon = \sqrt{\frac{\Sigma w_x(m_x - m'_x)^2}{n-3}} = \sqrt{\frac{81{\cdot}597}{62}} = \pm 1{\cdot}1339.$$

In the solution previously made, when $\log q$ was taken $= {\cdot}04$, ϵ was $\pm 1{\cdot}1304$. These results would lead us to infer (if no mistake has been made in the computation) that the assumption $\log q = {\cdot}04$ introduced no error appreciable by the observations, and that we may safely adopt that value as the best attainable in this case.

It has before been remarked that the experience of even the oldest American companies is insufficient to supply trustworthy information as to the mortality prevailing in extreme old age. Nevertheless, it may sometimes be desirable, for certain purposes, to extend the scale from the point where the observations fail, to the limit of human life, according to our general knowledge, from other sources, of the mortality for this period. As to the precise mode of connecting the observed rates with those according to a known table, adopted as a standard, without rendering the transition too violent, no general rules can be laid down. A process which will, I think, answer the purpose sufficiently well in most cases, is the following:

Comparing the rates of mortality deduced from the observations with those of the standard table at the older ages, where, on account of the paucity of the data, we begin to lose confidence in the former, and incline to the latter as the properer index of the probabilities of dying; we can conceive of a point where we shall be unable to assign a preference for either. Denote the observed rates by single accents and those of the standard table by double accents. Put, also, for the age x at which these two measures of the probability of dying may be presumed of equal weight,

$$m^{\circ}_x = \tfrac{1}{2}(m'_x + m''_x).$$

Then, taking convenient equidistant intervals on either side of the age x, we can obtain for any age within the limits $x-nw$ and $x+nw$ the interpolated value m°_{x+t} from the system of differences derived from the values m'_{x-nw}, $m'_{x-(n-1)w}$, ..., m'_{x-w}, m°_{x}, m''_{x+w}, ..., m''_{x+nw}. In general, it will not be worth while to employ more than five terms, in which case $n=2$.

Denote by a and c the means of the first and of the third differences immediately above and below the line of m°_{x}; and by b and d the second and fourth differences on this line. Then for the age $x+t$ we shall have, by the theory of interpolation,

$$m^{\circ}_{x+t}=m^{\circ}_{x}+\left(\frac{t}{w}\right)a+\left(\frac{t}{w}\right)^2\frac{1}{1.2}b+\left(\frac{t}{w}+1\right)\left(\frac{t}{w}\right)\left(\frac{t}{w}-1\right)\frac{1}{1.2.3}c+\left(\frac{t}{w}+1\right)\left(\frac{t}{w}\right)^2\left(\frac{t}{w}-1\right)\frac{1}{1.2.3.4}d$$

which may be directly employed. I have found it more convenient, however, to use this series developed according to the powers of t, putting

$$\left.\begin{aligned}A&=\frac{1}{w}(a-\tfrac{1}{6}c)\\ B&=\frac{1}{w^2}\frac{1}{1.2}(b-\tfrac{1}{12}d)\\ C&=\frac{1}{w^3}\frac{1}{1.2.3}c\\ D&=\frac{1}{w^4}\frac{1}{1.2.3.4}d\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (12)$$

whence we have

$$m^{\circ}_{x+t}=m^{\circ}_{x}+At+Bt^2+Ct^3+Dt^4 \quad . \quad . \quad . \quad (13)$$

t being taken positive for older, and negative for younger, ages than x.

A check on the whole computation is that at ages $x\pm 2w$, $x\pm w$, the interpolation should reproduce the fundamental values.

For illustration, let us in the previous example suppose that it is desired to extend the series in column (8) to the end of life, hypothetically, making use of the Actuaries' (old) table for this purpose, according to the foregoing method.

Take $x=70$, $w=5$. Then

$$\begin{aligned}m'_{x}&=\cdot 0503933\\ m''_{x}&=\cdot 0649329\\ \hline 2)&\cdot 1153262\\ \hline m^{\circ}_{x}&=\cdot 0576631\end{aligned}$$

and the values of the function and the system of differences will be as given in the following schedule:

x	m	Δ_1	Δ_2	Δ_3	Δ_4
60	·0245351				
		+·0100036			
65	·0345387		+·0131208		
		·0231244		+·0016519	
70	·0576631		·0147727		−·0094756
		·0378971		−·0078237	
75	·0955602		·0069490		
		·0448461			
80	·1404063				

whence

$$\left.\begin{aligned} a &= +\cdot030{,}510{,}75 \\ b &= +\cdot014{,}772{,}7 \\ c &= -\cdot003{,}085{,}9 \\ d &= -\cdot009{,}475{,}6 \end{aligned}\right\} \text{give by (12)} \left\{\begin{aligned} A &= +\cdot006{,}205{,}0 \\ B &= +\cdot000{,}311{,}246{,}6 \\ C &= -\cdot000{,}004{,}114{,}53 \\ D &= -\cdot000{,}000{,}631{,}707 \end{aligned}\right.$$

and with these values of the coefficients in (13) we obtain the hypothetical rates of mortality in column (10), between ages 60 and 80. Above this point the rates are those of the Actuaries' table.

From the rates of mortality in column (8) up to age 60, and those in column (10) from 60 to the end of life, I have computed the numbers living and dying [columns (11) and (12)], starting with a radix of one million persons at age 10. The irregularities in the second differences at ages 60 and 81 might have been considerably reduced, but I did not think this worth while, as my design was simply illustration, and not the deduction of the best numerical results in this particular instance. And I wish further to state, in conclusion, that the same consideration must be my apology, if any be needed, for whatever errors may have crept into the preceding computations, as well as for some details which have reference simply to the numerical conduct of the example. It is to the general method itself that I would invite criticism.

Reduction of Formulæ for Annuities and Assurances investigated by Mr. Sprague, on the common hypothesis of equal decrements in each year of life. *By* W. S. B. Woolhouse, F.R.A.S., F.S.S., &c.

IN 1869 Mr. Sprague read before the Institute an interesting paper "On the value of Reversionary Annuities payable half-yearly, quarterly, &c., according to the conditions which prevail in practice." This valuable paper is given in the *Journal*, vol. xv, page 126. The conditions which prevail in practice, as here for the first time introduced and mathematically investigated by Mr. Sprague, are the

following:—For a yearly reversionary annuity on a life x after the death of another life y, the first payment of the annuity is supposed to be made just one year after the day of the death of y, provided that x be then living; and for a like annuity payable by m instalments in each year, the first payment of an instalment is supposed to be made just $\frac{1}{m}$th of a year after the day of the death of y. Also, in each case, it is further supposed that whenever x shall die, after surviving y, the annuity shall be completed up to the day of the death of x by the supplementary payment of the proportion of the annuity estimated in respect of the fractional interval that may then have elapsed since the last yearly payment or instalment was received.

In the latter, and, perhaps, the most interesting portion of his paper, Mr. Sprague has, with his usual accuracy and ability, investigated a mathematical expression for the present value of the annuity of practice, on the usual assumption of equal decrements in each year of life; and, according to the same hypothesis, he has also appended expressions for what I have, in a former paper, denominated continuous assurances, that is, assurances payable at the instant of decease. Each of these expressions as they now stand, is, however, so prolix as to be wholly unfit for ordinary calculation, and Mr. Sprague himself states that the formula for the value of the complete reversionary annuity payable yearly, the most simple case, is so complicated as to be practically useless. In the present article it is proposed to examine the particular constitution of these formulæ, and the special results derivable from them, for the purpose of finally reducing the several expressions to more convenient and practical forms. For greater clearness, the several processes will be given *in extenso*.

To begin with the formulæ for assurances, I shall take the present opportunity of introducing and elucidating the use of a new quantity, which, it appears to me, ought henceforth to be admitted as a recognised element of assurance investigations appertaining to two lives, viz., the value of an assurance payable at the end of a year, and which is contingent upon both lives failing in that year. As the contingency here stated is of the second order, the value of an assurance of this kind, which, for brevity, may be appropriately termed a *Conjoint Assurance*, must always be a small quantity.

The probability of the life x failing in the nth year, is $p_{x,n-1}-p_{x,n}=\frac{p_{x-1,n}}{p_{x-1}}-p_{x,n}$; the same in respect to the life y, is

$\frac{p_{y-1,n}}{p_{y-1}}-p_{y,n}$; and if W_{xy} denote the present value of the conjoint assurance on the two lives x and y, then

$$\begin{aligned}
W_{xy}&=\Sigma v^n\Big(\frac{p_{x-1,n}}{p_{x-1}}-p_{x,n}\Big)\Big(\frac{p_{y-1,n}}{p_{y-1}}-p_{y,n}\Big)\\
&=\Sigma v^n\Big(\frac{p_{x-1.y-1,n}}{p_{x-1.y-1}}+p_{xy,n}-\frac{p_{x-1.y,n}}{p_{x-1}}-\frac{p_{x.y-1,n}}{p_{y-1}}\Big)\\
&=\frac{a_{x-1.y-1}}{p_{x-1.y-1}}+a_{xy}-\frac{a_{x-1.y}}{p_{x-1}}-\frac{a_{x.y-1}}{p_{y-1}}\\
&=\frac{1+a_{xy}}{1+i}+a_{xy}-\frac{a_{x-1.y}}{p_{x-1}}-\frac{a_{x.y-1}}{p_{y-1}}\\
&=\frac{1+(2+i)a_{xy}}{1+i}-\frac{a_{x-1.y}}{p_{x-1}}-\frac{a_{x.y-1}}{p_{y-1}}\quad \ldots\ldots \quad (1)
\end{aligned}$$

From this and the well known forms

$$\left.\begin{aligned}
A^{1}_{xy}&=\frac{1}{2}\Big(\frac{1-ia_{xy}}{1+i}+\frac{a_{x-1.y}}{p_{x-1}}-\frac{a_{x.y-1}}{p_{y-1}}\Big)\\
A_{x\overset{1}{y}}&=\frac{1}{2}\Big(\frac{1-ia_{xy}}{1+i}-\frac{a_{x-1.y}}{p_{x-1}}+\frac{a_{x.y-1}}{p_{y-1}}\Big)
\end{aligned}\right\}\quad \ldots \quad (2)$$

the following are obtained, viz.,

$$\left.\begin{aligned}
\frac{a_{x-1.y}}{p_{x-1}}-a_{xy}&=A^{1}_{xy}-\frac{1}{2}W_{xy}\\
\frac{a_{x.y-1}}{p_{y-1}}-a_{xy}&=A_{x\overset{1}{y}}-\frac{1}{2}W_{xy}
\end{aligned}\right\}\quad \ldots\ldots \quad (3)$$

$$\left.\begin{aligned}
\frac{1+a_{xy}}{1+i}-\frac{a_{x.y-1}}{p_{y-1}}&=A^{1}_{xy}+\frac{1}{2}W_{xy}\\
\frac{1+a_{xy}}{1+i}-\frac{a_{x-1.y}}{p_{x-1}}&=A_{x\overset{1}{y}}+\frac{1}{2}W_{xy}
\end{aligned}\right\}\quad \ldots \quad (4)$$

Referring now to Mr. Sprague's paper, at page 137 he gives for the value of a contingent assurance payable at the instant of the death of x provided y survive, on the supposition of equal decrements, the formula $\overline{A}^{1}_{xy}=$

$$\frac{i\delta-i+\delta}{(1+i)\delta^2}+\frac{2i\delta-2i+2\delta-i^2}{(1+i)\delta^2}a_{xy}-\frac{i\delta-i+\delta}{\delta^2}\cdot\frac{a_{x.y-1}}{p_{y-1}}+\frac{i-\delta}{\delta^2}\frac{a_{x-1.y}}{p_{x-1}}\quad \ldots \quad (5)$$

When the life y is replaced by certainty, the fraction $\frac{a_{x-1.y}}{p_{x-1}}$ becomes $\frac{a_{x-1}}{p_{x-1}}$, or $\frac{1+a_x}{1+i}$; and hence, in this particular case, (5) gives

$$\bar{A}_x=\frac{i\delta-i+\delta}{(1+i)\delta^2}+\frac{2i\delta-2i+2\delta-i^2}{(1+i)\delta^2}a_x-\frac{i\delta-i+\delta}{\delta^2}a_x+\frac{i-\delta}{\delta^2}\cdot\frac{1+a_x}{1+i}$$

$$=\frac{i\delta}{(1+i)\delta^2}+\frac{2i\delta-i+\delta-i^2}{(1+i)\delta^2}a_x-\frac{i\delta-i+\delta}{\delta^2}a_x$$

$$=\frac{i}{(1+i)\delta}-\frac{i^2}{(1+i)\delta}a_x=\frac{i}{\delta}\cdot\frac{1-ia_x}{1+i}=\frac{i}{\delta}A_x \quad . \quad . \quad . \quad (6)$$

Again, from (2) and (5) we obtain generally,

$$\delta\bar{A}^1_{xy}-iA^1_{xy}=\frac{i\delta-2i+2\delta}{2(1+i)\delta}+\frac{4i\delta-4i+4\delta-2i^2+i^2\delta}{2(1+i)\delta}a_{xy}$$

$$-\frac{i\delta-2i+2\delta}{2\delta}\cdot\frac{a_{x.y-1}}{p_{y-1}}-\frac{i\delta-2i+2\delta}{2\delta}\cdot\frac{a_{x-1.y}}{p_{x-1}}$$

$$=\frac{i\delta-2i+2\delta}{2\delta}\left\{\frac{1+(2+i)a_{xy}}{1+i}-\frac{a_{x.y-1}}{p_{y-1}}-\frac{a_{x-1.y}}{p_{x-1}}\right\}$$

$$=\frac{i\delta-2i+2\delta}{2\delta}W_{xy}.$$

Here we may observe that the coefficient

$$\frac{i\delta-2i+2\delta}{2\delta}=i\left(\frac{1}{2}-\frac{1}{\delta}+\frac{1}{i}\right)=i\left(\frac{1}{2}-\frac{1}{\delta}+\frac{1}{e^\delta-1}\right)=i\left(\frac{\delta}{12}-\frac{\delta^3}{720}\cdots\right)$$

is, with great practical precision, represented by $\frac{i\delta}{12}$, since the terms rejected cannot affect even the seventh place of decimals. Hence, as W_{xy} is a symmetrical function as regards x and y, there will result the two relations,

$$\left.\begin{aligned}\delta\bar{A}^1_{xy}-iA^1_{xy}&=\frac{i\delta}{12}W_{xy}\\ \delta\bar{A}_{\overset{1}{x}y}-iA_{\overset{1}{x}y}&=\frac{i\delta}{12}W_{xy}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

By addition and subtraction, these give also,

$$\delta\bar{A}_{xy}-iA_{xy}=\frac{i\delta}{6}W_{xy} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\delta(\bar{A}^1_{xy}-\bar{A}_{\overset{1}{x}y})=i(A^1_{xy}-A_{\overset{1}{x}y}) \quad . \quad . \quad . \quad . \quad (9)$$

The reductions here made in the formulæ for assurances will now enable us to reduce Mr. Sprague's complicated formula for the complete reversionary annuity, according to the conditions which prevail in practice. Before proceeding, however, we may observe that, whenever recourse is had to the relations (7) and (8), the exact coefficient $\frac{i\delta-2i+2\delta}{2\delta}$ will be retained in place of $\frac{i\delta}{12}$. Thus

no error whatever will be permitted to enter beyond what belongs to the hypothesis, and any simplification of an approximate nature will be deferred to a final step.

The expression investigated by Mr. Sprague for the present value of the annuity when payable yearly, and incomplete, is the following, viz.,

$$\frac{i^2}{(1+i)\delta^2}a_x-\frac{i\delta-i+\delta}{(1+i)\delta^2}a_{xy}-\frac{i-\delta}{\delta^2}\cdot\frac{a_{x.y-1}}{p_{y-1}} \quad . \quad . \quad (10)$$

For the correction of this formula, in order to make the annuity complete—that is, for the present value of the addition to be made on account of the payment of the annuity up to the day of death—Mr. Sprague determines separately two component portions, viz.,

$$\frac{2i-\delta^2-2\delta}{2(1+i)\delta^3}\left\{1-ia_x-(1+i)\left(\frac{a_{x-1.y}}{p_{x-1}}-a_{xy}\right)\right\} \quad . \quad . \quad (11)$$

$$\frac{(1+i)\delta^2+2\delta-2i}{2(1+i)\delta^3}\left\{-ia_x-a_{xy}+(1+i)\frac{a_{x.y-1}}{p_{y-1}}\right\} \quad . \quad . \quad (12)$$

the sum of which gives, for the required correction, the expression,

$$\frac{2i-\delta^2-2\delta}{2(1+i)\delta^3}-\frac{i^2}{2(1+i)\delta}a_x+\frac{(2i-\delta^2-2\delta)(2+i)-i\delta^2}{2(1+i)\delta^3}a_{xy}$$
$$-\frac{2i-\delta^2-2\delta}{2\delta^3}\cdot\frac{a_{x-1.y}}{p_{x-1}}+\frac{(1+i)\delta^2+2\delta-2i}{2\delta^3}\cdot\frac{a_{x.y-1}}{p_{y-1}} \quad . \quad . \quad . \quad (13)$$

Adding (10) to (13), Mr. Sprague's formula for the value of the complete reversionary annuity payable yearly is therefore

$$\frac{2i-\delta^2-2\delta}{2(1+i)\delta^3}+\frac{(2-\delta)i^2}{2(1+i)\delta^2}a_x+\frac{2i-2\delta+i^2-2i\delta^2-2\delta^2}{(1+i)\delta^3}a_{xy}$$
$$-\frac{2i-\delta^2-2\delta}{2\delta^3}\cdot\frac{a_{x-1.y}}{p_{x-1}}+\frac{(3+i)\delta^2+2\delta-2i-2i\delta}{2\delta^3}\cdot\frac{a_{x.y-1}}{p_{y-1}} \quad . \quad . \quad (14)$$

To effect a reduction of this expression, we proceed to reduce separately the component expressions (10), (11), and (12), with the aid of the foregoing results. The formula (10) reduces as follows, viz.,

$$(10) \quad =\frac{i}{\delta^2}\left(\frac{1-ia_{xy}}{1+i}-\frac{1-ia_x}{1+i}\right)-\frac{i-\delta}{\delta^2}\left(\frac{a_{x.y-1}}{p_{y-1}}-a_{xy}\right)$$
$$=\frac{i}{\delta^2}(\mathrm{A}_{xy}-\mathrm{A}_x)-\frac{i-\delta}{\delta^2}\left(\mathrm{A}_{\frac{1}{xy}}-\frac{1}{2}\mathrm{W}_{xy}\right)$$
$$=\frac{i}{\delta^2}\left(\frac{\delta}{i}\overline{\mathrm{A}}_{xy}-\frac{i\delta-2i+2\delta}{i\delta}\mathrm{W}_{xy}-\frac{\delta}{i}\overline{\mathrm{A}}_x\right)$$
$$-\frac{i-\delta}{\delta^2}\left(\frac{\delta}{i}\overline{\mathrm{A}}_{\frac{1}{xy}}-\frac{i\delta-2i+2\delta}{2i\delta}\mathrm{W}_{xy}-\frac{1}{2}\mathrm{W}_{xy}\right)$$

$$=\frac{1}{\delta}(\overline{A}_{xy}-\overline{A}_x)-\left(\frac{1}{\delta}-\frac{1}{i}\right)\overline{A}_{\underset{xy}{1}}+\left(\frac{i-\delta}{2\delta^2}-\frac{i+\delta}{\delta^2}\cdot\frac{i\delta-2i+2\delta}{2i\delta}\right)W_{xy}$$

$$=\frac{1}{\delta}\{(1-\delta\bar{a}_{xy})-(1-\delta\bar{a}_x)\}-\left(\frac{1}{\delta}-\frac{1}{i}\right)\overline{A}_{\underset{xy}{1}}+\frac{i^2-i\delta^2-\delta^2}{i\delta^3}W_{xy}$$

$$=\bar{a}_x-\bar{a}_{xy}-\left(\frac{1}{\delta}-\frac{1}{i}\right)\overline{A}_{\underset{xy}{1}}+\frac{i^2-i\delta^2-\delta^2}{i\delta^3}W_{xy}\quad .\quad .\quad .\quad .\quad (15)$$

Again, the value of the first part of the correction of the annuity is

$$(11)\quad =\frac{2i-\delta^2-2\delta}{2\delta^3}\left\{\frac{1-ia_x}{1+i}-\left(\frac{a_{x-1.y}}{p_{x-1}}-a_{xy}\right)\right\}$$

$$=\frac{2i-\delta^2-2\delta}{2\delta^3}\left(A_x-A_{\underset{xy}{1}}+\frac{1}{2}W_{xy}\right)\quad .\quad .\quad .\quad .\quad (16)$$

And the second part of the correction is

$$(12)\quad =\frac{(1+i)\delta^2+2\delta-2i}{2\delta^3}\left\{\frac{1-ia_x}{1+i}-\left(\frac{1+a_{xy}}{1+i}-\frac{a_{x.y-1}}{p_{y-1}}\right)\right\}$$

$$=\frac{(1+i)\delta^2+2\delta-2i}{2\delta^3}\left(A_x-A_{\underset{xy}{1}}-\frac{1}{2}W_{xy}\right)\quad .\quad .\quad (17)$$

The sum of (16) and (17) gives the total correction of the annuity, viz.,

$$(13)\quad =\frac{i}{2\delta}(A_x-A_{\underset{xy}{1}})+\frac{4i-2\delta^2-4\delta-i\delta^2}{4\delta^3}W_{xy}$$

$$=\frac{i}{2\delta}\left(\frac{\delta}{i}\overline{A}_x-\frac{\delta}{i}\overline{A}_{\underset{xy}{1}}+\frac{i\delta-2i+2\delta}{2i\delta}W_{xy}\right)+\frac{4i-2\delta^2-4\delta-i\delta^2}{4\delta^3}W_{xy}$$

$$=\frac{1}{2}(\overline{A}_x-\overline{A}_{xy}+\overline{A}_{\underset{xy}{1}})-\frac{i\delta-2i+2\delta}{2\delta^3}W_{xy}$$

$$=\frac{1}{2}\{(1-\delta\bar{a}_x)-(1-\delta\bar{a}_{xy})\}+\frac{1}{2}\overline{A}_{\underset{xy}{1}}-\frac{i\delta-2i+2\delta}{2\delta^3}W_{xy}$$

$$=-\frac{\delta}{2}(\bar{a}_x-\bar{a}_{xy})+\frac{1}{2}\overline{A}_{\underset{xy}{1}}-\frac{i\delta-2i+2\delta}{2\delta^3}W_{xy}\quad .\quad .\quad .\quad .\quad .\quad (18)$$

Adding this correction (18) to the annuity (15) we get finally for the present value of the complete reversionary annuity payable yearly, the following expressions, viz.,

$$(14)\quad =\frac{i}{\delta}\left(\frac{1}{\delta}-\frac{1}{2}\right)(A_{xy}-A_x)+\frac{i\delta-2i+2\delta}{2\delta^2}A_{\underset{xy}{1}}$$

$$+\left(\frac{2i-\delta^2-2\delta}{2\delta^3}-\frac{i\delta-2i+2\delta}{4\delta^2}\right)W_{xy}$$

$$\text{or}\quad =\left(\frac{1}{\delta}-\frac{1}{2}\right)(\overline{A}_{xy}-\overline{A}_x)+\frac{i\delta-2i+2\delta}{2i\delta}\overline{A}_{\underset{xy}{1}}$$

$$+\left(\frac{i^2-i\delta^2-\delta^2}{i\delta^3}-\frac{i\delta-2i+2\delta}{2\delta^3}\right)W_{xy}$$

$$\text{or} \quad = \left(1-\frac{\delta}{2}\right)(\bar{a}_x-\bar{a}_{xy})+\frac{i\delta-2i+2\delta}{2i\delta}\bar{\text{A}}_{\overset{1}{xy}}$$

$$+\left(\frac{i^2-i\delta^2-\delta^2}{i\delta^3}-\frac{i\delta-2i+2\delta}{2\delta^3}\right)\text{W}_{xy} \quad . \; . \; . \quad (19)$$

Here, by substitution of $i=e^\delta-1$, and expansion, the last term is

$$\left(\frac{i^2-i\delta^2-\delta^2}{i\delta^3}-\frac{i\delta-2i+2\delta}{2\delta^3}\right)\text{W}_{xy}=-\frac{i\delta}{360}\left(1+\frac{\delta^2}{21}\;\cdots\cdots\right)\text{W}_{xy},$$

the value of which is practically quite insensible, and may therefore be rejected.

By other approximate substitutions which in practice may be considered as exact, the formulæ for the complete reversionary annuity (19) become ultimately

$$(14) \quad = \frac{i}{\delta}\left(\frac{1}{\delta}-\frac{1}{2}\right)(\text{A}_{xy}-\text{A}_x)+\frac{i}{12}\text{A}_{\overset{1}{xy}}+\frac{1}{6}\text{W}_{xy}$$

$$\text{or} \quad = \left(\frac{1}{\delta}-\frac{1}{2}\right)(\bar{\text{A}}_{xy}-\bar{\text{A}}_x)+\frac{\delta}{12}\bar{\text{A}}_{\overset{1}{xy}}\,.$$

$$\text{or} \quad = \left(1-\frac{\delta}{2}\right)(\bar{a}_x-\bar{a}_{xy})+\frac{\delta}{12}\bar{\text{A}}_{\overset{1}{xy}} \quad . \; . \; . \; . \quad (20)$$

which last agrees in form with the general expression determined, by a most simple and comprehensive process, in my paper on "An Improved Theory of Annuities and Assurances,"* where (*Journal*, vol. xv, page 113) it is shown, generally, that the value of the complete reversionary annuity of practice, when payable in m instalments in each year, is

$$\left(1-\frac{\delta}{2m}\right)(\bar{a}_x-\bar{a}_{xy})+\frac{\delta}{12m^2}\bar{\text{A}}_{\overset{1}{xy}} \quad . \; . \; . \quad (21)$$

This last formula is derived by a general method, without any supposition of equal decrements in each year of life, or indeed any supposition whatever affecting the law of mortality. The perfect identity in the form of the resulting expression, when the annuity is payable yearly, must not be understood to imply an agreement as to absolute value. The values of the continuous quantities $\bar{a}_x$, $\bar{a}_{xy}$, $\bar{\text{A}}_{\overset{1}{xy}}$, &c., which enter the formulæ (20), are severally affected by the imperfection of the hypothesis of equal decrements, and through them the corresponding small error of the annuity is imparted to the numerical result. The quantities which exclusively depend upon yearly intervals, viz., a_x, a_{xy}, A_x, &c., are, however, independent of the hypothesis, but the formula for the annuity, expressed in terms of them, involves the value of the conjoint assurance and differs from the true formula in form as well as in value.

* It is my intention shortly to draw up for publication a treatise, embracing a full development of this important theory, and its practical applications.

The errors due to the hypothesis of equal decrements are readily found from what precedes. For example, to find the errors of $\overline{A}_x$ and $\bar{a}_x$, we have (Improved Theory, formula (29)),

$$\overline{A}_x=\delta\left(\frac{1}{i}-a_x+\frac{\mu_x}{12}\right)=\delta\left(\frac{A_x}{d}+\frac{\mu_x}{12}\right).$$

According to the hypothesis in question, the value is $\frac{i}{\delta}A_x$. Therefore the error in the value of the continuous assurance $\overline{A}_x$ is

$$\frac{i}{\delta}A_x-\delta\left(\frac{A_x}{d}+\frac{\mu_x}{12}\right)=\left(\frac{i}{\delta}-\frac{\delta}{d}\right)A_x-\delta\frac{\mu_x}{12}$$
$$=\frac{i\delta}{12}A_x-\delta\frac{\mu_x}{12}=\frac{\delta}{12}(iA_x-\mu_x)$$

And since $\bar{a}_x=\frac{1}{\delta}(1-\overline{A}_x)$, the error in the value of the continuous annuity $\bar{a}_x$ is $\frac{1}{12}(\mu_x-iA_x)$. The errors of the joint and survivorship assurances $\overline{A}_{xy}$, $\overline{A}_{\overset{1}{xy}}$, since they depend on the relations (7) and (8), will involve the value of the conjoint assurance W_{xy}.

From the reductions that have been so far accomplished, we conclude, generally, that the assumption of equal decrements in each year of age, the hypothesis usually adopted, for convenience or expediency, in treatises on annuities and assurances, although very interesting and suggestive as a subject of discussion, does not yield us any additional facility of calculation as a set off against the small errors and imperfections that are necessarily engendered in so many of the quantities that come under our consideration.

On Mathematical Statistics and its application to Political Economy and Insurance. By Dr. Theodor Wittstein, *Actuary of the Hanover Life Insurance Company. Translated by* T. B. Sprague, M.A.

PART I.

Introduction.

Concerning a New Science, being a Paper read before the Mathematico-Physical Section of the 40th Natural Philosophy Congress, held at Hanover in 1865.

It is well known that the number of the physical sciences is continually increasing, by a process of development from within. We constantly see divisions and branches of existing sciences

cultivated and developed until they acquire a separate existence as new and independent sciences. For it has long been impossible for a single person to master the whole, or even a considerable part of these sciences, and the necessity for a division of labour is continually becoming more and more felt here, as elsewhere. It is, however, comparatively seldom that the number of the sciences is increased by the addition to them of an entirely new one, and therefore the case of which I am about to speak appears to deserve our particular attention. For I shall presently prove that a department of science, which has not hitherto belonged to the physical sciences, is now being so developed and perfected, that in a very short time it will claim admission into the number on equal terms with the others. This department of science is *Statistics*. The science of Statistics in its present form has now existed a full century. It owed its birth, as well as its name, to Achenwall of Göttingen, about the middle of the last century. It may be defined as the compilation and comparison of all the noteworthy facts that a State—or, in more general terms, a Society—presents to our view at a given epoch; of what Schlözer designates by a single word "*Staatsmerkwürdigkeiten.*" But to this definition the subsidiary idea soon attached, that the information in question must be given principally in numbers; and thus we see that the present text books on statistics really consist, for the most part, of a collection of tables giving all the information that can be expressed in figures, as to the population, manufactures, agriculture, trade, &c., &c., of a country. In this form, statistics is one of the social sciences, and is especially considered as an important auxiliary to the science of political economy. Now we must bear in mind that, philosophically considered, a collection of tables containing observed facts can make no claim to be considered a science in the proper sense of the word. Such tables only furnish the rough materials from which the science is to be constructed. In scientific phrase, they are only a series of *observations,* this very word expressing what is the next step required. The problem before us is the same as in all other branches of natural philosophy—to ascend from observations to the discovery of natural laws. In this direction, statistics, faithful to its definition, has hitherto done scarcely anything. It has, in fact, only drawn the obvious conclusions which result from a first glance at the tables, and, therefore, when it attempts to solve the above problem, it becomes essentially a new science. It has to begin at the point where statistics, as hitherto understood, ceases; and since the new science deals principally with figures, we must

employ mathematics for the solution of the problem. The new science may therefore be called *Mathematical Statistics,* unless mathematicians should prefer to call it *Analytical Statistics,* according to the analogy of Analytical Optics, Analytical Mechanics, &c.

This will be made still clearer by a comparison with another instance of which the circumstances are more generally known. The astronomer makes observations and collects them in the form of tables; but if he regarded his work as then completed, astronomy could never have claimed to be called a science. It would have remained simply in the stage occupied by statistics as hitherto understood. Such nearly was the position of astronomy when Tycho Brahe made his observations on the planet Mars, which have since become so celebrated. Then came Kepler, who from these observations deduced his well known laws. In Kästner's happy phrase, "Tycho quarried the marble, and Kepler chiseled the statue." It was this step, afterwards completed by Newton's theoretical investigations, that first raised astronomy to the rank of a science, properly so called; and this step has not yet been made by statistics. It must not, however, be supposed that a Kepler only is required to raise statistics to the dignity of a science. Its Tycho has not yet appeared; for statistical facts and figures are as yet, with scarcely an exception, of very doubtful value, and (as I shall presently show) unsuited for mathematical investigation. Nay, keeping to the same metaphor, it may be said that its Copernicus is still to come, who shall for the first time give us a general view of the facts we ought to observe. Statistics is now in its infancy—at the same stage as astronomy was when it was nothing but astrology and drew horoscopes. It still happens every day, that people prove by statistical figures whatever they choose, the only thing necessary for the purpose being a kind of dexterity, happily expressed in the French phrase, "*grouper les nombres.*"* This will be amended as soon as mathematics, with its inexorable logic, shall have mastered the statistical materials. The calculus of probabilities is the special branch that must here be applied, and that not in the sense so commonly attached to the phrase by statisticians, according to which it is a calculus that gives *inexact* results as distinguished from exact calculations, but that ingenious science which is as completely based on logic as any part of

* Say gives one of many instances in his *Traité d'économie politique*: "The French Minister of the Interior, in his report for 1813, a time of calamity, when commerce was destroyed and the national resources of every kind were rapidly diminishing, boasts of having demonstrated by means of figures that France was in a condition of prosperity greater than it had ever before enjoyed."

mathematics, and which has been brought to so splendid a development by Laplace and Gauss. Only thus will statistics grow to be a science, in the full meaning of the word; and only thus will really useful and trustworthy results be obtained. It may even be predicted that, in a future century, Mathematical Statistics will solve problems of the bare enunciation of which we have as yet no idea.

What has been here said of statistics in general, is not equally true of all the divisions of the science; for it is only the statistics of population which now manifestly admits of mathematical treatment, whereas the other divisions must for the present be excluded from it. In the statistics of population some steps have indeed been already made, which may be considered as an attempt to lay the foundation of mathematical statistics, but these steps are so far from satisfactory, and have made so little use of the means of investigation which analysis in its present advanced condition could have supplied, that it is necessary to disregard them and commence the investigation anew from the very beginning.

The first and most important idea with which the mathematical treatment of the statistics of a population has to deal, is that of mortality; for this enters into every question that can be proposed with regard to a population. For the purpose of answering enquiries as to the mortality that will prevail among a given population or any other given body of persons, it has been usual to form what is known as a mortality table, which represents the rate at which that body of persons will die off, on the supposition that the rate of mortality prevailing at one instant of time will remain unchanged thereafter. A large number of such mortality tables are to be found collected in statistical text books, but they are all, more or less, of doubtful value. Various attempts have also been made to find a formula that shall represent the number living at any age in the table; but these attempts were certain to fail, so long as the fundamental facts were deficient in the accuracy which, as stated above, was characteristic of Tycho's observations. Even the function, first proposed by Gompertz and since discussed by so many persons, $y=a^{b^x}$, which leads in its application to logarithmic integrals and gamma functions, must remain a very doubtful hypothesis until a sufficiently trustworthy collection of facts can be obtained to test its accuracy.

It is, indeed, no exaggeration to assert that all existing mortality tables, without exception, are far from being so trustworthy as could be desired. This is owing, partly to the imperfection of

the facts on which they are based, partly to the imperfection of the methods hitherto employed in their construction. As regards the former, the censuses in Germany are at present taken in so unsatisfactory a manner, that calculations based on them can only be considered as rough approximations to the truth. This is, at all events, true as regards the censuses within the limits of the Zollverein, whereas in Belgium and France, where to be sure they go to greater expense, matters appear to be in a better condition. Having elsewhere stated my views as to what science requires in a census, I will not repeat them here,* but will only add that, unfortunately, the public are but little disposed to assist in the matter. It is, for instance, well known that women universally state their ages too low; nay, at the last census in Hanover, two ladies positively refused to state their ages at all, so that the enumeraters could do nothing but guess at them. Under such circumstances, we must, perhaps, give up all idea of ever obtaining a completely trustworthy census. More accurate data can be obtained from the experience of Life Insurance Companies, Annuity Companies, Widows' Funds, and similar societies, but hitherto very few of these have been at the pains to extract and arrange, with a view to strict mathematical treatment, the statistical facts lying buried in their records; and we therefore possess much fewer mortality tables derived from such sources than might have been expected. But to tables of this kind the second of the above mentioned objections applies; for they are defective, in consequence of the defective methods employed in their construction. Even the two which at the present time are considered the best, and are extensively used, namely, Brune's Table, deduced from the experience of the General Widows' Fund, in Berlin, and that deduced from the experience of the 17 English Offices, are open to this objection; for the methods employed in their construction, so far as they are known, are far from being theoretically correct.

It must be admitted that the labour bestowed by statisticians on the formation of mortality tables is so far justified, that a mortality table presents in a very clear and intelligible form the facts to which it relates, and exhibits extremely well the instantaneous changes in a population. But, after all, it is only a popular form, insufficient for scientific purposes, and it has accordingly been forcibly pointed out by other writers † that the formation of a

* See my Essay, *Zur Bevolkerungs-Statistik*, in the *Zeitschrift des Königlich Preussischen statistischen Bureau*, 3rd year, Part I.

† See, especially, Fischer's "*Grundzüge des auf die menschliche Sterblichkeit gegründeten Versicherungswesens*," Oppenheim, 1860.

mortality table cannot be the first step of scientific enquiry. The fundamental idea from which our treatment of the statistics of a population must start, is rather that of the *probability of dying,* or more exactly, *the probability that a person who belongs to a definite group will die within a year.* Statistics has hitherto lacked this idea, because, as already remarked, it has not had a correct idea of the calculus of probabilities in general; and with the introduction of the above idea, the science becomes essentially a new one. We can, if we please, substitute for the above probability, that of living a year, which is the complement to unity of it. If either the one or the other of these probabilities is known for every year of age, it requires but little labour to deduce a mortality table.

The analytical expedients required for the solution of the problem have been so fully developed by Laplace, in his *Théorie Analytique des Probabilités,* that it is rather surprising they have not been earlier applied to it. It must be confessed, however, that it is but lately that the want has become fully felt, in consequence of various recent writers having attempted the solution of the questions under consideration. These have certainly, one after the other, made steps towards it, but have not yet reached the desired goal. We shall here endeavour, with the help of the above mentioned expedients, to make another step forward.

If it has been observed that out of a group of L persons, there are L′ alive at the end of a year, and there have been no entries into this group, nor withdrawals from it during the year (in which cases a special investigation would be required), it has been the custom hitherto to say that the probability, p, of living a year is given for each person of this group by the equation

$$p=\frac{L'}{L}.$$

But it is not forgotten to add that if, under the like circumstances, other values of L and L′ were given, the value of p may generally be different. Hence it logically follows that the calculation gives the value of p only inexactly; and thus originates the strange idea prevailing among the unlearned that the calculus of probabilities gives inexact results. Now, in reality, the above form of expression is wholly incorrect. The value of p given by the above equation makes no claim whatever to be the *true value* of the unknown probability, but it is the *most probable value* resulting from the observations that have been made. Now to the most probable value there always belongs a *probable error,* and a *mean*

error, either of which may be used to measure the degree of trustworthiness of that most probable value. This probable and this mean error have, however, never yet been determined.

Again, if the probability p of living a year is given, and we wish, under the same circumstances as before, to know the number, l', which will be alive at the end of a year out of a new group of l persons, it has hitherto been the custom to say that it is given by the equation,

$$l'=lp.$$

At the same time it is admitted, that actual observation may show a number different from l', so that in this case too the calculation may be charged with being inexact. But here, also, the usual form of expression, as given above, is inaccurate; for the value of l' given by the equation is not the *true value*, but the *most probable value* of the number alive at the end of a year. Hence this value also has a *probable error*, and a *mean error*, either of which measures its degree of trustworthiness; and these probable and mean errors have likewise never yet been determined.

These two cases furnish examples of a few of the gaps which statistics, as hitherto understood, has in vain sought to fill up. To fill them up, and to build further on the ground thus gained, is the object of the following investigations, which will be found to take an entirely new direction. The degree of precision in our treatment of the subject, and the important applications it admits of, will be seen so directly from the investigations themselves, that it appears unnecessary to say more of them here.

PART II.

General Investigations as to Mortality and Mortality Tables.

Chapter 1.

The probability of living a year is supposed to be given.

§ 1.

Let p be the probability that a person of the age x will be alive at the end of a year.* This is most commonly given by means of

* It may be remarked, in passing, that the following investigations admit of being applied to many other subjects besides mortality. Thus, for instance, we might have said, "Let p be the probability that an unmarried person of the age x will still be unmarried at the end of a year;" or "Let p be the probability that a building of class x will not be burnt down at the end of a year," and so on; the nature of the particular subject indicating what modifications must be introduced in applying our investigations to such cases as these. We will not pursue this subject further on the present occasion.

a mortality table; the ratios of the numbers living at successive ages being equal to the different values of this probability. Then, according to well known principles, the probability that out of l persons of the age x, l' will be alive at the end of a year, is

$$P=\frac{\underline{|l}}{\underline{|l'}\,\underline{|l-l'}}p^{l'}(1-p)^{l-l'} \quad . \quad . \quad . \quad . \quad (1)$$

Here l' can have any integral value from 0 to l inclusive.

The value of l' which makes P a maximum can be found by comparing the above value of P with the values it receives when $l'-1$ and $l'+1$ are substituted for l'. It is given by the relations,

$$\frac{l-l'+1}{l'}>\frac{1-p}{p}>\frac{l-l'}{l'+1};$$

i.e.

$$\frac{l'+1}{l+1}>p>\frac{l'}{l+1};$$

whence, if l is a large number, and l_0 denotes the most probable value of l', we have

$$l_0=lp\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the number given in a mortality table as living at the age $x+1$, if l is the number living at the age x.

§ 2.

Putting $l'=l_0+k=lp+k$, then by (1) the probability that the number alive at the end of a year will exceed by k the most probable number, is

$$P=\frac{\underline{|l}}{\underline{|lp+k}\,\underline{|l(1-p)-k}}\cdot p^{lp+k}.(1-p)^{l(1-p)-k} \quad . \quad . \quad (3)$$

Here k may have any integral value from $-lp$ to $l(1-p)$.

If we assume that l, $lp+k$, $l(1-p)-k$ are large numbers, and apply Stirling's formula

$$\underline{|n}=\sqrt{2\pi}.n^{n+\frac{1}{2}}.\epsilon^{-n}, \text{ approximately,}$$

equation (3) becomes

$$P=\frac{1}{\sqrt{2\pi}}\cdot\frac{l^{l+\frac{1}{2}}}{(lp+k)^{lp+k+\frac{1}{2}}\{l(1-p)-k\}^{l(1-p)-k+\frac{1}{2}}}\cdot p^{lp+k}\cdot(1-p)^{l(1-p)-k}$$

$$=\frac{1}{\sqrt{2lp(1-p)\pi}}\cdot\frac{1}{\sqrt{\left(1+\frac{k}{lp}\right)\left\{1-\frac{k}{l(1-p)}\right\}}}\cdot$$

$$\frac{1}{\left(1+\frac{k}{lp}\right)^{lp+k}\left\{1-\frac{k}{l(1-p)}\right\}^{l(1-p)-k}}\cdot$$

Expanding the second factor by the Binomial Theorem, it becomes

$$1+\frac{(2p-1)k}{2lp(1-p)}+\ldots.$$

and the Naperian logarithm of the third factor is

$$-(lp+k)\left(\frac{k}{lp}-\tfrac{1}{2}\frac{k^2}{l^2p^2}+\ldots.\right)$$
$$-\{l(1-p)-k\}\left\{-\frac{k}{l(1-p)}-\tfrac{1}{2}\frac{k^2}{l^2(1-p)^2}-\ldots.\right\}$$

or

$$-\frac{k^2}{2lp(1-p)}+\ldots.$$

the other terms containing powers of l in the denominator, and therefore vanishing in comparison with the first.

If, then, for brevity, we put

$$\frac{1}{2lp(1-p)}=h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the above formula becomes

$$\mathrm{P}=\frac{h}{\sqrt{\pi}}\{1+(2p-1)h^2k\}\epsilon^{-h^2k^2} \quad . \quad . \quad . \quad (5)$$

or finally, since we consider k to be small in comparison with l,

$$\mathrm{P}=\frac{h}{\sqrt{\pi}}\cdot\epsilon^{-h^2k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It may be noticed, that it hence follows, since the extreme values of which k is capable extend from $-\infty$ to $+\infty$, that

$$\int_{-\infty}^{\infty}\mathrm{P}dk=1.$$

Observation. The following numerical examples will enable the reader to judge of the accuracy of Stirling's formula applied above.

n	Log $\lfloor n$		Error.
	By actual Calculation.	By the Formula.	
5	2·07918	2·07195	+·00723
10	6·55976	6·55615	·00361
25	25·19065	25·18920	·00145
50	64·48307	64·48235	·00072
100	157·97000	157·96964	·00036

From this it is seen that even for values of n that are not large, the error of the five-figure logarithm affects only such part

of it as we should consider of doubtful accuracy on account of the uncertainty of our data. But it would be a wholly needless attempt at exactness to use more than 5 places of decimals in calculations of this kind.

§ 3.

The probability that the number living at the end of a year will lie between $lp+k$ and $lp-k$ is by (6)

$$=\frac{2h}{\sqrt{\pi}}\int_0^k \epsilon^{-h^2k^2}dk \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If this probability is to be equal to $\frac{1}{2}$, and σ is the value of k for which this happens, we have, as is well known,*

$$\sigma=\frac{\cdot 6745}{h\sqrt{2}};$$

whence, by means of (4),

$$\sigma=\cdot 6745\sqrt{lp(1-p)} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and this is the *probable error* to be expected in the observed number l'; or, in other words, it is an even bet that the number living at the end of a year will lie between the limits

$$lp\pm\cdot 6745\sqrt{lp(1-p)}.$$

The quantity $\frac{1}{h\sqrt{2}}$, which is here $=\sqrt{lp(1-p)}$, represents, as is well known, the *mean error* of the observations.

It follows that, the mortality being the same, the probable error in the number living at the end of a year is proportional to the square root of the number observed. On the contrary, the ratio $\frac{\sigma}{l}$, that is, the ratio that the probable error bears to the number observed, is inversely proportional to the square root of the number observed, and can therefore be made as small as we choose, by taking that number sufficiently large. These conclusions hold also for the mean error, and for every other characteristic error. The probable error in the number living at the end of a year is evidently identical with the probable error in the number dying within the year.

§ 4.

If l_0, the most probable number living at the end of a year, is known beforehand, as is the case in calculations based upon a given

* See Gauss's *Theoria combinationis observationum.* The coefficient ·6745 is more exactly ·6744897, and its five-figure logarithm is $\bar{1}\cdot 82898$.

mortality table (see the end of § 1), the equation (8) can be put in the form

$$\sigma = \cdot 6745 \sqrt{\frac{l_0(l-l_0)}{l}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

that is, it is an even chance that the number living at the end of a year will lie between the limits

$$l_0 \pm \cdot 6745 \sqrt{\frac{l_0(l-l_0)}{l}}.$$

For instance, according to Brune's table, out of 7943 males of the age 40, 7847 are alive at the end of a year. If we put these numbers respectively for l and l_0 in the above formula, we get $\sigma = 6{\cdot}57$; and it is an even chance that if 7943 men of the age 40 are under observation, the number living at the end of a year will lie between 7840·43 and 7853·57.

Observation. If the probability (7) is to have the values ·9, ·99, ·999, ·9999, the coefficient ·6745 in (8) and (9) is replaced by 1·6449, 2·5758, 3·2918, 3·8906, respectively. (See Gauss, *Theoria comb. obs.*) Consequently the odds are 9 to 1 that the number living at the end of a year will lie between the limits

$$lp \pm 1{\cdot}6449 \sqrt{lp(1-p)},$$

or

$$l_0 \pm 1{\cdot}6449 \sqrt{\frac{l_0(l-l_0)}{l}}.$$

In other words, it is to be expected in a large number of observations that, in every 10 cases in which l such persons are observed, the number living at the end of a year will only once fall outside these limits. Similarly in the other cases.

§ 5.

The foregoing calculation is not confined to a single year, but can be at once applied to any term of years for which the value of p is given. In particular, starting from the number living at any age we choose, in a given table of mortality, we can immediately calculate the probable error in the number living at any higher age, by substituting for l_0 in formula (9) the number living at that age according to the table.

For instance, taking Brune's table for males, in which 7943 is the number living at the age 40, we get the following probable errors in the number living at various ages.

Age.	Number Living.	Probable Error in the Number Living.	
		Absolute.	Percentage of Number Living.
40	7943		
50	6845	21	·3
60	5304	28	·5
70	3100	29	·9
80	954	20	2·1
90	46	5	10·8

The probable error in the number living attains its maximum value when $l_0 = \frac{l}{2}$, that is, for the probable duration of life (in the above table for the age of 66, for which $\sigma = 30$); and beyond that point diminishes down to 0. On the contrary, the ratio $\frac{\sigma}{l_0}$, or the ratio that the probable error in the number living bears to that number, continually increases.

(*To be continued.*)

The Law of Life Insurance in France as affected by a recent decision of the Supreme Court of Judicature. By M. Leon de Montluc, A.I.A., *Avocat à la Cour de Paris.*

NEVER was a more complete change suddenly brought about in the laws of a nation by legislative enactment than that which has taken place this year in France in the law of life insurance, in consequence of one single decision of the Supreme Court of Judicature. Up to the present time the construction given to the contract of life insurance in this country has been quite different from what it is in England. As there is no provision of written law that relates to life insurance, it being not even so much as mentioned in the Civil or Commercial Codes, people thought themselves justified in governing it by laws and rules of their own. For instance, although it is a principle of law common to both English and French jurisprudence (we may add, to the law of all legislating nations from time immemorial) that *choses in action* shall necessarily devolve upon our legal representatives after our death, it has hitherto been decided almost universally by French tribunals that an exception was to be made in favour of

life insurance policies. By the advocates of that doctrine, the right in the sum assured was thought never to have vested in the person effecting the policy, and the assurance monies were said to be transferred directly, *i. e., omisso medio,* from the assurer to the party entitled to receive the sum assured; and that sum, accordingly, would not be liable to succession duty.

To us that doctrine always appeared quite untenable; we could not regard the assurance monies as standing on a different footing from any other portion of the assets a person may be possessed of, and though we were not actually confident of the ultimate success of our opinion, we set forth, in a book written in 1867, the reasons that made us believe that the time would arrive when the decisions of so many Courts, conformable to the doctrine of most people versed either in the law or the practice of life insurance, would be over-ruled some day by the *Supreme Court.* We triumphed, at last, beyond our most sanguine expectations: the *outlawry* of Life insurance is now no more; it will henceforward be subject, as any other contract, to the operation of the written law. It may be true that the interest of the public exchequer, which was a loser in the positions maintained by our adversaries, contributed a little in turning the scale of victory; but, be that as it may, the construction of life insurance is now restored to sounder principles. Contrary to the doctrine of the Courts of Caen, Lyon, Colmar, Rouen, Paris, it has been held by the *Court of Cassation* (*Chambre Civile*) that the assurance monies form part of the estate of the deceased, like any other description of property. It was an action brought by the Stamp Officers (*L'Enregistrement v. Krieg*) for the payment of an *ad valorem* duty upon a sum of 20,000 francs insured by one M. Krieg for the benefit of his legal representatives, and made payable within four months after his death. The policy had been disposed of by M. Krieg in his will, but the legatee not having accepted it, the legal representative of the deceased became entitled to its benefit. The tribunal of *Saverne* decided for the defendant (21st May 1869), upon the ground that the benefit of a life assurance policy was not deemed to confer a succession; but its decision has been annulled by the above mentioned judgment of the *Court of Cassation,* which is to the following effect:—

"Whereas it appeared that Krieg insured upon his life at the *Caisse Générale des Familles,* in consideration of an annual premium of 420 francs, the sum of 20,000 francs, made payable to his representatives within four months after his death;

"And that the right in the said sum of 20,000 francs, corresponding to the premiums to be paid, combined with the aleatory value of the risk assured, formed a part of the assets of the insured, who could dispose, and did dispose of it, and that, the legatee not having accepted the bequest, it devolved upon his heirs, who found it in his estate;

"And that consequently the said sum ought to have been included by the defendants in the return to be made of the succession, and was liable to the succession duty;

"And that the decree brought into Court, having decided in the contrary, has violated the above mentioned provisions of the law;

"Quashes it." (7 February 1872.)

So far as jurisprudence goes, this decision of the Court of Cassation is quite unexceptionable, for it has laid down the only principles conformable to the present state of French legislation; but we do not mean to say that the law ought not to be altered. We know life insurance deserves to be encouraged by the government, and in France needs such encouragement, and we should have no objection to any legislative amendment that would either make a general exemption from the succession duty in favour of life insurance policies, or, at least, exempt from it all policies, under a certain sum, made for the benefit of one's wife, husband, or children.

Note on Mr. Woolhouse's paper "On the Philosophy of Statistics."

MR. D. Carment has pointed out to us a close verbal agreement between a passage in Mr. Woolhouse's paper "On the Philosophy of Statistics" (see p. 37 of this volume), and Mr. O. G. Downes's translation of M. Quetelet's *Letters on the Theory of Probabilities.* It appears, indeed, that certain portions of pages 40, 41, and 42, have been transcribed from the latter work with but little alteration. Mr. Woolhouse states, with reference to this, that "the object "sought to be accomplished in the paper is that of embodying a "full and comprehensive view of what has been done on a subject "not generally understood, and at the same time of making the "discussion as complete as possible by giving much original and "important matter not to be found elsewhere in form or substance." Considering the wide range of the subject, and its intimate connection with an important branch of mathematical science, entire

originality is of course not to be expected in a full account of it; and the fact of Mr. Woolhouse having, in his general introductory statements, borrowed more or less from M. Quetelet's dissertations, simply amounts to an admission that the latter had expressed himself so happily with regard to the matters the former had under discussion, that he did not think he could improve on the phraseology. It should also be borne in mind, that the letters of M. Quetelet are not to be regarded exclusively as an original source of information, but chiefly as an intelligible and familiar exposition of the scientific results of his predecessors.—Ed. *J. I. A.*

Correction of an Error in Milne's Treatise on Annuities and Assurances.

We are indebted to Mr. Henry Hoskins, of 28 Notting Hill Square, for the following:—

In a footnote on pp. 254-256, Mr. Milne has investigated the value of a joint-life annuity on the hypothesis of equal decrements throughout life, but for the latter part of the investigation, commencing at the 5th line of the footnote on p. 255, should be substituted:—

Thus it appears that the total value of the annuity is

$$\mathrm{U}\left(1+\frac{u}{mm'(1-u)}\right)+\mathrm{W}\cdot\frac{\dfrac{2u}{1-u}-(m+m')}{mm'}-\frac{(m+1)^2u^{m+1}-u}{mm'\pi(1-u)},$$

which expression by writing $\dfrac{\mathrm{U}}{1-u}-\dfrac{mu^{m+1}}{\pi(1-u)}$ for W, and observing that $\mathrm{U}=\dfrac{u(1-u^m)}{\pi(1-u)}$, may be transformed to

$$\frac{u}{1-u}\left[\frac{1}{\pi}-\frac{1}{m'u}\left(\frac{2u}{\pi(1-u)}\cdot u^m-\left\{\frac{2u}{1-u}+1-(m+m')\right\}\frac{\mathrm{U}}{m}\right)\right]$$

$$=\frac{\dfrac{1}{\pi}-\dfrac{1}{m'}\cdot\left(1+\dfrac{r}{\nu}\right)^{\mu}\cdot\left[\dfrac{2}{\pi\left\{\left(1+\frac{r}{\nu}\right)^{\mu}-1\right\}}\cdot\left(1+\dfrac{r}{\nu}\right)^{-m\mu}-\left\{\dfrac{2}{\left(1+\frac{r}{\nu}\right)^{\mu}-1}+1-(m+m')\right\}\dfrac{\mathrm{U}}{m}\right]}{\left(1+\dfrac{r}{\nu}\right)^{\mu}-1}$$

Cor. 1. When the annuity is payable at the same ν equal intervals in the year at which the interest is convertible, ν being $=\pi$, and $\mu=1$, the formula last obtained becomes

$$\frac{1}{r}\left[1-\frac{v\left(1+\frac{r}{v}\right)}{m'}\left(\frac{2}{r}\cdot\left(1+\frac{r}{v}\right)^{-m}-\left\{\frac{2v}{r}+1-(m+m')\right\}\frac{\text{U}}{m}\right)\right]$$

Cor. 2. When the expectation of the senior of the two lives is s years and that of the junior j years, $m=2vs$, $m'=2vj$, and the formula becomes

$$\frac{1}{r}\left[1-\frac{1+\frac{r}{v}}{j}\left(\frac{1}{r}\cdot\left(1+\frac{r}{v}\right)^{-2vs}-\left\{\frac{1}{r}+\frac{1}{2v}-(s+j)\right\}\frac{\text{V}}{2s}\right)\right]$$

V being the value of an annuity certain of one pound for the term of $2s$ years.

Cor. 3. When the interest and the annuity are each payable but once a year, $v=1$, and the value of the annuity is

$$\frac{1}{r}\left[1-\frac{1+r}{j}\left(\frac{1}{r}\cdot(1+r)^{-2s}-\left\{\frac{1}{r}+\frac{1}{2}-(s+j)\right\}\frac{\text{V}}{2s}\right)\right].$$

The Life Assurance Companies Acts, 1871 *and* 1872.

34 & 35 Vict. Chap. 58. A.D. 1871.

An Act to amend the Life Assurance Companies Act, 1870. [24th July 1871.]

WHEREAS by section three of the Life Assurance Companies Act, 1870, a sum of money is required in certain cases to be deposited with the Accountant General of the Court of Chancery, to be invested and returned by him in manner therein directed, and it is expedient to make further provision in respect of the deposit, investment, and return of such sum: 33 & 34 Vict. c. 61.

Be it therefore enacted by the Queen's most Excellent Majesty, by and with the advice and consent of the Lords Spiritual and Temporal, and Commons, in this present Parliament assembled, and by the authority of the same, as follows:—

Payment into court and orders as to sums deposited under 33 & 34 Vict. c. 61, s. 3.

1. Every sum required by the Life Assurance Companies Act, 1870, to be deposited with the Accountant General of the Court of Chancery shall be paid into the Court of Chancery, and orders with respect to the payment of such money into and out of court, and the investment and return thereof, and the payment of the dividends and interest thereof, may be from time to time made, altered, and revoked by the like authority and in the like manner as orders with respect to the payment into and

out of court, and the investment of other money, and the application of the dividends and interest thereof.

Amendment of section 25 of 33 & 34 Vict. c. 61.

2. Section twenty-five of the Life Assurance Companies Act, 1870, shall be construed as if the words "chapter twenty-four" were and had at and from the date of the passing of such last-mentioned Act been inserted therein in place of "chapter forty-one;" and Her Majesty's Printers shall in all copies of the Life Assurance Companies Act, 1870, which may be printed after the passing of this Act, insert the words "chapter twenty-four" in the place of the words "chapter forty-one" in section twenty-five of the said Life Assurance Companies Act, 1870.

Construction and short title.

3. This Act shall be construed as one with the Life Assurance Companies Act, 1870, and that Act and this Act may be cited together as The Life Assurance Companies Acts, 1870 and 1871, and this Act may be cited as The Life Assurance Companies Act, 1871.

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35 & 36 Vict. Chap. 41. A.D. 1872.

An Act to amend the Life Assurance Companies Acts, 1870 and 1871. [6th August 1872.]

BE it enacted by the Queen's most Excellent Majesty, by and with the advice and consent of the Lords Spiritual and Temporal, and Commons, in this present Parliament assembled, and by the authority of the same, as follows:

Deposit by Company in Court of Chancery.

1. Whereas by the provisions of the "Life Assurance Companies Acts, 1870 and 1871," a life assurance company is required to pay a sum of money into the Court of Chancery by way of deposit, and the certificate of incorporation of such company is not to be issued unless such deposit has been made, and such deposit is to be returned to the company as soon as its life assurance fund amounts to the sum therein mentioned; and doubts have arisen as to the construction of the said provisions, and it is expedient to remove such doubts; be it therefore enacted as follows:

The said deposit may be made by the subscribers of the memorandum of association of the company, or any of them, in the name of the proposed company, and such deposit upon the incorporation of the company shall be deemed to have been made by and to be part of the assets of the company.

The said deposit shall, until returned to the company, be deemed to form part of the life assurance fund of the company, and shall be subject to the provisions of section four of the Life Assurance Companies Act, 1870, accordingly. The Board of Trade may from time to time make, and when

made revoke, alter, or add to, rules with respect to the payment and repayment of the said deposit, the investment of or dealing with the same, the deposit of stocks or securities in lieu of money, and the payment of the interest or dividends from time to time accruing due on any such investment, stocks, or securities in respect of such deposit. Any rules made in pursuance of this section shall have effect as if they were enacted in this Act, and shall be laid before Parliament within three weeks after they are made, if Parliament be then sitting, or if not, within three weeks after the beginning of the then next session of Parliament.

Separation of life funds.

2. Whereas, by section four of the Life Assurance Companies Act, 1870, it is enacted that, "In the case of a "company established after the passing of this Act, trans-"acting other business besides that of life assurance, a "separate account shall be kept of all receipts in respect "of the life assurance and annuity contracts of the com-"pany, and the said receipts shall be carried to and form "a separate fund, to be called the life assurance fund of "the company, and such fund shall be as absolutely the "security of the life policy and annuity holders as though "it belonged to a company carrying on no other business "than that of life assurance, and shall not be liable for "any contracts of the company for which it would not "have been liable had the business of the company been "only that of life assurance;" and further provisions were made by the same section, with respect to the application of the above recited part of the said section to existing companies, and doubts have arisen with respect to the construction of the said provisions, and it is expedient to remove such doubts; be it therefore enacted,

That the portion of section four of the Life Assurance Companies Act, 1870, above recited shall apply to every company established before the passing of that Act, provided that the Life Assurance Companies Act, 1870, and this Act shall not diminish the liability of the life assurance fund for any contracts of the company entered into before the passing of the Life Assurance Companies Act, 1870.

Deposit of statement and abstract required by 33 & 34 Vict. c. 61, s. 10.

3. Whereas by section ten of the Life Assurance Companies Act, 1870, it is provided that, "Every annual "statement so deposited after the next investigation shall "be accompanied by a printed copy of the abstract re-"quired to be made by section seven," be it therefore enacted that the words "next investigation" shall be construed to mean the first investigation after the passing of the said Act.

The Board of Trade shall lay before Parliament any statement or abstract of report which is deposited with them by any company, and purports to be in pursuance of the Life Assurance Companies Act, 1870, although the

Board are of opinion that it is not such a statement or abstract as is required to be prepared by that Act.

Winding up of subsidiary company to be ancillary to winding up of principal company.

4. Where the business or any part of the business of a life assurance company has, either before or after the passing of this Act, been transferred to another company under an arrangement in pursuance of which such first-mentioned company (in this Act called the subsidiary company) or the creditors thereof has or have claims against the company to which such transfer was made (in this Act called the principal company), then, if such principal company is being wound up by or under the supervision of the court, either at or after the passing of this Act, the court shall (subject as herein-after mentioned) order the subsidiary company to be wound up in conjunction with the principal company, and may by the same or any subsequent order appoint the same person to be liquidator for the two companies, and make provision for such other matters as may seem to the court necessary, with a view to such companies being wound up as if they were one company; and the commencement of the winding up of the principal company shall, save as otherwise ordered by the court, be the commencement of the winding up of the subsidiary company; the court nevertheless shall have regard, in adjusting the rights and liabilities of the members of the several companies between themselves, to the constitution of such companies, and to the arrangements entered into between the said companies, in the same manner as the court has regard to the rights and liabilities of different classes of contributories in the case of the winding up of a single company, or as near thereto as circumstances admit.

Where any subsidiary company or company alleged to be subsidiary is not in process of being wound up at the same time as the principal company to which it is subsidiary, the court shall not direct such subsidiary company to be wound up unless after hearing all objections (if any) that may be urged by or on behalf of such company against its being wound up, the court is of opinion that such company is subsidiary to the principal company, and that the winding up of such company in conjunction with the principal company is just and equitable.

Where any subsidiary company and principal company are being wound up by different branches of the court, the court to which appeals from such branches lie shall make an order directing in which branch the winding up of such companies is to be carried on, and the necessary proceedings shall be taken for carrying such order into effect.

An application may be made in relation to the winding up of any subsidiary company in conjunction with a principal company by any creditor of, or person interested in, such principal or subsidiary company.

Where a company stands in the relation of a principal

company to one company, and in the relation of a subsidiary company to some other company, or where there are several companies standing in the relation of subsidiary companies to one principal company, the court may deal with any number of such companies together or in separate groups, as it thinks most expedient, upon the principles laid down in this section.

Valuation of annuities and policies.

5. Where a life assurance company is being wound up by the court, or subject to the supervision of the court, or voluntarily, the value of every life annuity and life policy requiring to be valued in such winding up shall be estimated in manner provided by the First Schedule to this Act, but this section shall not apply to any company the winding up of which has commenced before the passing of this Act, unless the court having cognizance of the winding up so order, which order that court is hereby empowered to make, if it think it expedient so to do, on the application of any person interested in the winding up of such company.

Rules in First and Second Schedules to be rules of court.

6. The rules in the First and Second Schedules to this Act shall be of the same force as if they were rules made in pursuance of the one hundred and seventieth, one hundred and seventy-first, and one hundred and seventy-third sections of "The Companies Act, 1862," as the case may be, and may be altered in manner provided by the said sections, and rules may be made under the said sections for the purpose of carrying into effect the provisions of this Act with respect to the winding up of companies.

Regulation as to novations by policy holders.

7. Where a company, either before or after the passing of this Act, has transferred its business to or been amalgamated with another company, no policy holder in the first-mentioned company who shall pay to the other company the premiums accruing due in respect of his policy shall by reason of any such payment made after the passing of this Act, or by reason of any other act done after the passing of this Act, be deemed to have abandoned any claim which he would have had against the first-mentioned company on due payment of premiums to such company, or to have accepted in lieu thereof the liability of the other company, unless such abandonment and acceptance have been signified by some writing signed by him or by his agent lawfully authorised.

Construction and short title.

8. This Act shall be construed as one with the Life Assurance Companies Acts, 1870 and 1871; and those Acts and this Act may be cited together as "The Life Assurance Companies Acts, 1870 to 1872;" and this Act may be cited as "The Life Assurance Companies Act, 1872."

First Schedule.

Rule for valuing an Annuity.

An annuity shall be valued according to the tables used by the company which granted such annuity at the time of granting the same, and where such tables cannot be ascertained or adopted to the satisfaction of the court, then according to the table known as the Government Annuities Experience Table, interest being reckoned at the rate of four per centum per annum.

Rule for valuing a Policy.

The value of the policy is to be the difference between the present value of the reversion in the sum assured on the decease of the life, including any bonus or addition thereto made before the commencement of the winding up and the present value of the future annual premiums.

In calculating such present values the rate of interest is to be assumed as being four per centum per annum, and the rate of mortality as that of the tables known as the seventeen offices experience tables.

The premium to be calculated is to be such premium as according to the said rate of interest and rate of mortality is sufficient to provide for the risk incurred by the office in issuing the policy, exclusive of any addition thereto for office expenses and other charges.

Second Schedule.

Where an assurance company is being wound up by the court or subject to the supervision of the court, the official liquidator in the case of all persons appearing by the books of the company to be entitled to or interested in policies granted by such company, for life assurance, endowment, annuity, or other payment, is to ascertain the value of such policies, and give notice of such value to such persons, and any person to whom notice is so given shall be bound by the value so ascertained unless he gives notice of his intention to dispute such value in manner and within a time to be prescribed by a rule or order of the court.

The following rules have been made by the Board of Trade for the purpose of carrying out the objects of the above Acts :—

Board of Trade Rules.

The Board of Trade, in pursuance of the powers conferred upon them under the first section of "The Life Assurance Companies Act, 1872," have made the following rules with respect to the payment into the Court of Chancery, and repayment of the deposit required to be made by a life assurance company, in pursuance of the provisions of the Life Assurance Companies Acts, 1870 and 1871; the investment of the deposit in securities, the deposit of stocks or securities in

lieu of money, and the payment of the interest or dividends from time to time accruing due on any such stocks or securities.

1. In these rules the term "the court" means the High Court of Chancery in England, and the word "company" means a company as the same is defined in the second section of the Life Assurance Companies Act, 1870.

2. Where any company is required in pursuance of "The Life Assurance Companies Acts, 1870 to 1872" to deposit the sum of twenty thousand pounds with the Accountant-General of the Court of Chancery, the said company, or the subscribers of the memorandum of association of the said company, or any of them, as the case may be, (in these rules referred to as the promoters) may make application to the Board of Trade for a warrant, and the Board of Trade may thereupon issue their warrant to the promoters for such payment into court, which warrant shall be a sufficient authority for the company or persons therein named to pay the money therein mentioned into the Bank of England, in the name and with the privity of the said Accountant-General, and for that officer to issue directions to such bank to receive the same, to be placed to his account there, ex parte the company mentioned in such warrant, according to the method (prescribed by statute or general rules or orders of court or otherwise,) for the time being in force respecting the payment of money into the said court, and without fee or reward.

Provided, that in lieu, wholly or in part, of the payment of money, the promoters may bring into court as a deposit an equivalent sum of bank annuities, or of any stocks, funds, or securities in which cash under the control of the court is for the time being permitted to be invested, or of Exchequer bills, (the value thereof being taken at the price at which the promoters originally purchased the same, as appearing by the broker's certificate of that purchase); and in that case the Board of Trade shall vary their warrant accordingly by directing the transfer or deposit of such amount of stocks, funds, securities, or Echequer bills by the persons therein named, into the name or to the account of the said Accountant-General in trust to attend the orders of the court ex parte the company mentioned in such warrant.

3. At any time when the office of the Accountant-General of the Court of Chancery is closed, a deposit under these rules may nevertheless be made, in the manner and subject to the regulations provided with respect to deposits by companies by section 88, of the Lands Clauses Consolidation Act, 1845.

4. Where money is so paid into the Court of Chancery, the court may, on the application of the company or the persons named in the warrant of the Board of Trade, or of the majority or survivors of such persons, order that the same be invested in such stocks, funds or securities as the applicants desire and the court thinks fit.

5. In the subsequent provisions of these rules the term "the deposit fund" means the money deposited, or the stocks, funds, or securities in which the same is invested, or the bank annuities, stocks, funds, securities, or Exchequer bills transferred or deposited, as the case may be, and the term "the depositors" means the company or persons named in the warrant of the Board of Trade authorising the

deposit, or the majority or survivors of those persons, their executors, administrators, or assigns.

6. The court shall, on the application of the depositors, order the deposit fund to be paid, transferred, or delivered out to the applicants, or as they direct, so soon as it is proved to the satisfaction of the court that the life assurance fund of the company, in respect of which the deposit is made, accumulated out of premiums paid to the said company amounts to the sum of forty thousand pounds.

7. The depositors shall be entitled to receive payment of the interest or dividends from time to time accruing on or in respect of the deposit while in court. And the court may, from time to time, on the application of the depositors, make such order as may seem fit respecting the payment of the interest or dividends accordingly.

8. The issuing in any case of any warrant or certificate relating to deposit or to the deposit fund, or any error in any such warrant or certificate, or in relation thereto, shall not make the Board of Trade, or the person signing the warrant or certificate on their behalf, in any manner liable for or in respect of the deposit fund, or the interest or dividends accruing on the same, or any part thereof respectively.

9. Any application under these rules to the Court of Chancery shall be made in a summary way by petition.

W. R. MALCOLM.

Board of Trade,
28th August 1872.

HOME AND FOREIGN INTELLIGENCE.

GERMANY—LIFE ASSURANCE BUSINESS IN 1870.

We have received from Herr Finanzrath G. Hopf of Gotha, whose recent death we notice with very great regret, a statement of the progress and position of Life Assurance Companies in Germany in the year 1870.

Since 1867, the date of our last notice of these Companies (*vide Journal*, vol. xv, p. 55), six Offices have been established; the total number being at present 41.

The new business effected in 1870 fell very far short of that done in the previous year. It amounted

In 1868	to	£10,500,495	on	88,346	lives or policies,
„ 1869	„	11,693,744	„	95,696	„
„ 1870	„	8,563,514	„	66,516	„

This great decrease was, of course, caused by the war with France, which commenced in July 1870; but it is shown that in most of the Companies, particularly those which were principally affected by the war, the new business had steadily increased during the first half of

the year, and amounted to more than double that of the second half. The assurances in force were,

At the end of	1868, 400,841	lives or policies, for	£54,099,669
„	1869, 456,702	„	61,012,744
„	1870, 474,074	„	63,692,169.

The income from premiums and interest, the claims, and the assurance funds for these three years, were as follows:—

Year.	Income.	Claims.		Assurance Funds.
		Persons.	Amounts.	
	£		£	£
1868	2,177,988	6159	804,677	7,200,572
1869	2,420,661	7011	870,139	8,115,903
1870	2,592,983	8128	1,041,492	8,997,420

It will be noticed that the claims for 1870 are heavy, being greater by 19·69 per-cent than those for 1869. It appears, however, from returns made by 26 Companies, that they are somewhat less than the probable amount estimated by the tables of mortality.

The new assurances in 1870, as compared with 1869, have decreased 26·76 per-cent; the assurances in force at the end of 1870 have increased 4·39 per-cent, and the assurance funds, after deducting claims and expenses, have increased 10·86 per-cent during the year. The average amount assured by each policy is £134. 10*s*., and the average amount of each claim £128.

Great differences of opinion arose with respect to the mode in which the war risks should be dealt with. A few Offices felt bound absolutely to refuse permission to the assured to take part in the war. Some Offices sought to meet the emergency by placing these cases in a class by themselves, and forming a fund out of which the claims were paid either wholly or in part, according to the extra premiums received and the amount of the fund. Others undertook the full risk at special rates, varying from 5 to 10 per-cent of the sum assured for combatants, and from $3\frac{1}{3}$ to $6\frac{2}{3}$ per-cent for non-combatants. They limited the amount assured on war risks to less than their usual maximum, and only took these risks on policies effected before the assured were called to active service.

D. A. B.

Abstract of Table showing the Business and Position of the German Life Assurance Companies in the Year 1870.

Company.	Established.	Assurances existing at beginning of the Year.	New Assurances.	Assurances existing at end of the Year.	Income.	Claims paid.	Expenses of Management. Actual	Expenses of Management. Per-Cent of Income.	Assurance Fund. Amount.	Assurance Fund. Surplus.
		£	£	£	£	£	£		£	£
Gotha	1827	9,297,930	733,371	9,724,057	452,865	198,300	21,705	4·79	2,461,380	476,697
Lübeck	1828	3,106,912	253,477	3,141,106	116,714*	58,216	16,143*	13·83	464,541	22,310
Leipzig	1830	2,513,785	377,886	2,734,057	117,666	47,072	11,662	9·91	486,172	80,235
Hanover	1830	409,571	32,200	420,672	17,535	11,086	2,109	12·03	72,919	6,059
Berlin	1836	2,356,876	257,196	2,438,034	116,637	58,097	12,315	10·56	640,862	90,255
Brunswick	1842	132,883	3,486	132,245	6,429*	2,978	?	?	55,689	?
Frankfort	1844	1,000,000*	139,000*	1,056,143*	45,000*	20,944	5,080	11·29	186,373	9,101
Janus (Hamburg)	1847	1,905,813	214,286*	1,856,824	72,714	33,814	6,866	9·44	260,063	13,403
Teutonia	1852	1,472,129	239,167	1,464,705	54,572*	23,576	10,943	...	99,039	None
Concordia	1853	3,796,486	341,654	3,932,500	151,429*	62,930	?	?	698,359	143,311
Schwerin	1853	283,014	40,171	295,243	10,454	2,657	?	?	43,683	7,539
Iduna	1854	1,150,759	165,139	1,179,711	65,856	19,465	10,780	16·37	181,147	12,956
Magdeburg	1856	1,784,960	192,981	1,831,592	67,392	21,083	10,399	15·43	179,225	9,512
Thuringia	1856	1,542,578	125,580	1,529,686	49,304	19,788	6,857*	13·91	131,862	4,959
Germania	1857	7,018,259	699,626	6,852,955	250,435	92,546	31,890	12·73	635,249	22,678
Providentia	1857	1,018,182	116,972	1,025,907	35,053	15,474	6,571*	18·75	94,937	?
Railway	1861	744,219	150,993	780,361	28,039	8,155	5,431*	19·36	67,528	7,694
Prussian	1865	498,041	108,143*	533,299	30,708	5,467	7,574	24·66	31,734	2,169
Friedrich Wilhelm	1866	1,195,707	210,449	1,215,356	71,271	13,059	15,086	21·17	69,325	3,847
Nordstern	1867	652,511	108,061	688,409	29,243	4,679	5,551	18·98	55,234	3,825
Bremen	1867	26,495	19,529	42,692	1,457	219	328*	22·56	1,761	997
North German	1868	262,099	91,655	260,676	10,806	3,688	4,873	...	None	None
Potsdam	1868	171,167	186,572*	273,520	7,331	3,986	5,065	...	6,597	222
Munich	1836	456,208	29,898	457,617	17,949	11,902	?	?	94,841	11,577
Stuttgart	1854	2,459,983	308,615	2,686,905	112,794	30,599	?	?	412,186	105,101
Darmstadt Annuity Company	1855	133,791	12,233	140,081	5,933	2,143	?	?	28,073	3,877
Stuttgart "	1861	374,106	234,741	573,641	18,143*	4,496	?	?	39,475	6,856
Carlsruhe	1864	548,257	234,040	718,911	22,138	7,812	?	?	40,799	5,564
Janus (Vienna)	1839	762,883	206,246	908,479	34,924	16,142	?	?	126,612	26,397
Donau	1852	684,042	143,839	732,748	31,209	13,086	5,138	16·46	74,444	4,128
Anker	1858	2,295,364	312,328	2,401,350	90,000*	40,627	?	?	247,116	6,971
Patria	1866	263,991	127,169	316,685	13,560	5,132	4,264	31·44	20,832	1,607
Europa	1869	—	200,000*	145,252	5,571*	295	?	?	4,896	?
Various Offices (together)	—	7,571,429*	1,428,571*	7,857,143*	314,286*	135,714*	?	?	742,857*	?
Zürich	1857	1,409,351	175,383	1,492,911	52,286*	25,689	3,271*	6·26	137,196	30,329
Basel	1865	1,712,963	342,857	1,850,696	65,280	20,577	11,484	17·59	104,414	7,875
Total		61,012,744	8,563,514	63,692,169	2,592,983	1,041,493			8,997,420	

IMPERIAL LIFE INSURANCE COMPANY.

Established 1820.

QUINQUENNIAL REPORTS AND STATEMENT OF ACCOUNTS AS ON 31ST JANUARY, 1871.

REPORT OF THE DIRECTORS.

* * * * *

Two Balance Sheets are herewith submitted, one showing the state of the Proprietors' Fund and the other that of the Assurance and Annuity Funds. With regard to the first, it has not been thought desirable to alter the values of the various investments of which the Fund consists as they have heretofore stood in the Company's accounts, that value being the price at which they were severally bought; but in the Assurance Fund, the actual market value of the fluctuating securities as on the 31st January last has been adopted as a basis.

The Actuary's Report * * * shows a divisible surplus on the Assurance Fund of £63,750.

Of this one-fifth or £12,750 has to be appropriated to the Shareholders, and the Directors propose out of this sum to pay a Bonus of 30*s*. per Share with the Dividend in July next, which will absorb £11,250, and to add the remaining £1,500 to the Proprietors' Fund.

The four-fifths of the surplus belonging to the Assured will admit of a Reversionary Bonus of 10*s*. per cent. on sums Assured and previous Bonuses, for every complete Annual Premium paid since the 31st of January 1861, being added to Policies entitled to the benefits of quinquennial division, and 3*s*. per cent. on the sums assured to the few remaining Policies of the old decennial series for every year they shall have been in force. Bonuses at these rates are declared accordingly.

The additions to Policies thus made will vary from £8 15*s*. to 10*s*. per cent. according to duration, and the total Reversionary amount added will be £92,000.

The small surplus of £282 on the Annuity Fund will be carried forward.

The new sums Assured between 1861 and 1866 amounted to £671,000 and those between 1866 and 1871 to £811,800, which shows an increase of more than 20 per cent. * * * The Directors have seen with pleasure the passing of a legislative enactment which compels all Life Offices to make returns in prescribed forms to the Board of Trade, as they do not doubt the effect of this measure will be eventually to restore public confidence in all those offices which can show their position to be a sound one.

PROPRIETORS' FUND—31ST JANUARY, 1871.

LIABILITIES.

	£	s.	d.
Unclaimed Dividends	£1,606	4	0
Interest received since the declaration of last Dividend .	3,878	4	8
Balance of Fund	170,228	4	4
	£175,712	13	0

ASSETS.

£230	12	2	Three per Cent. Consols	£203	16	0
2,554	6	0	New Three per Cents	2,373	11	2
13,000	0	0	Metropolitan Railway Debentures	13,000	0	0
6,600	0	0	Victoria Government Railway Loan	6,600	0	0
25,000	0	0	Great Northern Railway Debentures	25,000	0	0
30,000	0	0	London and North-Western Railway Debentures	30,000	0	0
20,000	0	0	London, Brighton and South-Coast Railway Debentures	20,000	0	0
9,400	0	0	Great Eastern Railway B. Debenture Stock	9,877	15	4
8,986	15	8	Scinde, Punjaub and Delhi Railway Stock	9,150	0	0
20,000	0	0	British Guiana Bonds	20,850	0	0
20,000	0	0	City Bonds	20,225	0	0
8,000	0	0	Loan and Trust Company of Upper Canada	8,000	0	0
			Company's Shares (307)	4,944	13	9
			Cash at Messrs. Barclay and Co.'s (Dividend Account)	481	4	0
			Outstanding Interest	3	8	1
			Amount due, and to be transferred to this Fund	5,003	4	8
				£175,712	13	0

ASSURANCE AND ANNUITY FUNDS—31st JANUARY, 1871.

LIABILITIES.

Claims admitted and announced, but not paid				£42,106	5	0
Amount due and to be transferred to Proprietors' Fund				5,003	4	8
Rebate of Interest, paid in advance, on Loans on Policies				313	5	1
Premiums paid in advance				45	16	2
Policies surrendered before 31st January, but not then passed to Account				184	19	7
Annuity Fund	£14,937	11	7			
Interest accrued to this Fund, but not due	140	7	8			
				15,077	19	3
Assurance Fund				777,208	18	4
				£839,940	8	1

ASSETS.

			Mortgages on Property within the United Kingdom	£112,974	9	0
			Loans secured on Rates, Tolls, &c.	201,199	15	0
£75,000	0	0	New Three per Cents	68,331	12	2
25,000	0	0	Great Western Railway Debentures	25,000	0	0
25,000	0	0	North Staffordshire do.	25,000	0	0
27,000	0	0	Metropolitan do.	27,000	0	0
11,400	0	0	Great Indian Peninsula Railway Five per Cent. Debenture Stock	12,045	12	3
11,000	0	0	Great Indian Peninsula Railway Four per Cent. Debenture Stock	9,649	3	8
11,300	0	0	Madras Railway Debenture Stock	11,657	9	0
13,400	0	0	Victoria Government Railway Loan	13,613	17	0
10,000	0	0	Great Eastern Railway B. Debenture Stock	10,809	11	9
6,013	4	4	Scinde, Punjaub and Delhi Railway Stock	6,082	7	1
30,000	0	0	East Indian Railway Stock	27,766	5	4
22,500	0	0	London and St. Katherine's Dock Bonds	22,500	0	0
20,000	0	0	Hull Dock Bonds	20,000	0	0
15,000	0	0	London do.	15,000	0	0
10,000	0	0	Victoria Six per Cent. Bonds	11,052	16	0
5,700	0	0	New Zealand Five per Cent. Bonds	5,687	14	4
10,000	0	0	New South Wales Five per Cent. Bonds	10,009	11	9
4,000	0	0	New Brunswick Six per Cent. Bonds	4,175	15	2
10,000	0	0	New South Wales Government Loan of 1870	10,009	11	9
			Carried forward	£649,565	11	3

	£	s.	d.
Assets brought forward	£649,565	11	3
Loan and Trust Company of Upper Canada	2,000	0	0
Premises in Old Broad Street	20,000	0	0
Loans on the Company's Policies (within their surrender value)	117,277	2	3
Transfer Stamps in hand	29	1	0
Cash on Deposit and on Current Account	25,531	10	5
Bills Receivable	789	7	0
Petty Cash	66	11	8
Balances due by Agents	12,947	1	3
Outstanding Premiums, less re-assurances	2,196	13	1
Do. Interest	1,065	0	8
Interest on Investments accrued but not due	8,472	9	6
	£839,940	8	1

Actuary's Report.

On the 31st of January last, the fifth decennial period of the Imperial Life Insurance Company came to a close. By the Deed of Settlement under which the Company was constituted, it was provided that at the close of each decennial period an investigation should be made into its affairs, and that after such an amount of its funds had been set apart as would suffice to meet all its liabilities, the surplus, if any, should be divided between the Proprietors and the Policy-holders in the proportion of one-third to the former and two-thirds to the latter. Under the Act of Parliament obtained in the year 1841, the intervals of periodical investigation were reduced from ten years to five, and the Profits were directed to be divided in the ratio of one-fifth to the Proprietors and four-fifths to the Assured. The sixth of the quinquennial periods so instituted expired on the 31st of January.

Having completed a very careful valuation of the Company's liabilities as at that date, I beg to lay before you a statement of the results arrived at and of the principles on which the valuation has been conducted.

There were in force 3,516 Policies, by which, after deducting Re-assurances, a Sum of £2,768,485, inclusive of previous bonus additions, was Assured. Of these 2,705, assuring £2,095,662, bonus included, were entitled to participate in profits; the remaining 811, assuring £672,823, were non-participating. The total ordinary premium revenue derived from these Policies was £80,709.

Besides these Assurances the Company had contracted liabilities under Annuity Bonds, twenty in number, amounting to an annual sum of £1,744.

In valuing the assurance liabilities, with some inconsiderable exceptions presently to be noted, the table of mortality used was that known as the Seventeen Offices' Experience, and the rate of interest assumed was 4 per cent.

As the premiums paid by the Assured have to provide for profits and expenses of management as well as for the payment of the sums assured at maturity, it is of the utmost importance in valuing the future premium revenue that a due proportion of the same should be set apart as a **Liability**. If this be not done the portion assigned to meet Assured risks being unduly large will, by swelling the assets, give an appearance of prosperity which is not in accordance with fact. The principle which has governed my determination of the reserve for future profit and

expenses, in valuing the liabilities under Policies for the whole term of single life with and without participation in profits, is to set apart as "loading" the difference between the net 4 per cent. Offices' Experience premiums and the rate chargeable at the age when the Assurance was effected, according to the tables published in the present Prospectus of the Office. If the rates which are now charged had always been in use, this method would have been identical with what is technically styled "a net premium valuation," but the rates having been more than once altered, it does not come strictly within that category.

An examination of the Valuation Balance Sheet annexed to this Report will show that the loading reserved is upwards of 40 per cent. of the value of the net premium (that is of the gross premium less the loading) in the case of the participating Assurances, and about 17½ per cent. in that of the non-participating ones.

The exceptions to which I alluded above were the Joint Life, Joint Life and Survivor, Contingent and Endowment Assurances, which were valued by the net Carlisle 3 per cent. tables; the Assurances against Issue and the Endowments of Children, for which the full premium paid was reserved; and the Assurances for Short Terms, the reserve for which was estimated at three-fourths of the gross annual premium.

It should also be stated that a special reserve has been made for the Policies in the Java Agency. These Assurances, which amount to £96,630, are subject to a higher scale of premium than the home risks. When valued by any of the ordinary methods they would seem to require a smaller reserve, but as there is some difference of opinion as to the proper way of treating such cases, I have first valued them as though they were ordinary risks at home rates, and to the net liability so brought out I have added 30 per cent., by which means all question as to the sufficiency of the reserve is obviated. The amount involved is small and comparatively unimportant, but it is desirable that the most explicit information as to the way in which the results were reached should be given.

For the same reason I ought to mention that extra premiums charged for climate or occupation have not been included in the valuation. With regard to extra rates imposed for under-average health in the form of a definite per centage on the sum Assured, which was the method of treating such cases until May, 1867, these likewise have been excluded. The mode adopted after that date was to add a certain number of years to the actual age of the under-average life, and to charge the tabular rate for the age thus advanced. All these cases have been valued as if the advanced age were the true one. The result of the valuation of Assurance Risks thus carried out is to show a surplus of £67,800 in the Assurance Fund.

The Annuity Liabilities have been valued by Mr. A. G. Finlaison's Government Annuitants' Table (Report of 1860—observations No. 14 and 15) at 4 per cent. The resulting surplus is, as might be expected from the recent establishment of this branch of business, very small, and is carried over to the next quinquennium.

It has been found that on the basis of the Offices' Experience table at 4 per cent., the amount required to yield a Reversionary Bonus of 10*s*. per cent. on the sums Assured and previous bonuses for every premium upon Quinquennial Policies paid since the 31st of January, 1861, and of 3*s*. per cent. upon the few remaining Policies of the old

decennial series for every year they have been in force, is £51,000. If that amount be so appropriated out of the surplus of the Assurance Fund, the Proprietors, being entitled to one-fifth of the divisible surplus, should have a sum of £12,750 apportioned to them. These two amounts together account for £63,750, and there remains £4,050 to be reserved for contingencies.

VALUATION BALANCE SHEET—31ST JANUARY, 1871.

LIABILITIES.

Value of £2,102,752 Participating Sum Assured and Bonus (Whole Life)		£1,108,548
Value of £16,240 Premium Loading on the same		192,988
Value of £623,577 Non-participating Sum Assured (Whole Life)		291,934
Value of £3,025 Premium Loading on the same		38,963
Liability under assurances of sundry classes as under:—		
£20,729 Joint and Several Existence	£5,407	
£11,700 Joint Existence	1,946	
£31,575 Contingent Assurances	577	
£8,500 Endowment Assurances	452	
£9,100 Endowments	954	
£38,125 Short Period Assurances	741	
£29,650 Issue Cases	615	
£1,127 Absolute Reversion	120	
		10,812
Additional Reserve for Java Assurances		3,261
Value of £1,743 18*s*. 9*d*. per annum Immediate Annuities		14,796
Surplus of Annuity Fund carried to next Quinquennium		282
Surplus of Assurance Fund		67,800
		£1,729,384

ASSETS.

Value of £58,652 per annum Gross Premium on Participating Sum Assured	£666,269
Value of £717 do. Terminable Premiums on Participating Sum Assured	5,964
Value of £20,366 do. Gross Premium on Non-participating Sum Assured	232,085
Value of £2,385 do. Deferred Premiums on Non-participating Sum Assured	28,633
Value of £177 do. Terminable Premiums on Non-participating Sum Assured	1,070
Value of £10,200 Reassurances, Whole Life, With Profits	352
" £74,500 do. do. Without Profits	2,281
" £5,000 do. Short Period	70
" £18,650 do. Issue Cases	373
Annuity Fund	15,078
Assurance Fund	777,209
	£1,729,384

The following further particulars are extracted from the returns under the Fifth and Sixth Schedules of the Life Assurance Companies Act:—

By the Deed of Settlement under which the Company was constituted, it was directed "That the profits of the Company, after making ample provisions for satisfying all outstanding Claims and Liabilities shall, from time to time, be appropriated to increase the sums

assured by Policies for the whole term of Life (whether of Proprietors or not), and to increase the Shares of Proprietors in the Subscription Capital. And in order to regulate the appropriation of such profits, an account of the affairs of the Company and of all its property, effects, and out-standing engagements, shall be taken at the end of ten years from the General Court, which will be held in the month of January, One thousand eight hundred and twenty-one, and also at the end of every succeeding ten years. And as often as it shall appear, upon the taking of such periodical accounts, that there will be a surplus upon the Premium Account,* after providing for all existing or out-standing engagements (regard being had to the true value and extent thereof), then one-third of such surplus shall be added to the Subscription Capital. That the residue of such surplus shall be added by way of Bonus to the Policies of such persons (whether Proprietors or not) who shall, at the time the same shall be declared, have Assurances on foot with the Company upon and for the whole continuance of one or more Life or Lives, such Bonus to be in proportion to the amount insured by such Policy or Policies, and to the number of years during which such Policy or Policies shall have remained in force, but so as no person shall be entitled to any Bonus in respect of any Policy or Policies, unless he or she shall have paid five Annual Premiums thereon at the least."

By subsequent General Courts, exercising a power conferred on them by the Deed of Settlement, certain modifications of the above were introduced by RESOLUTION: that is to say, it was declared lawful for the Directors to accept surrenders of Bonuses or additions to Policies by payment of the value thereof, or by application of them to the reduction or extinction of future Premiums or otherwise; and also to grant Assurances upon the terms of excluding the assured from participation in profits.

By the terms of the Deed, the Proprietors were expressly precluded from making any alteration in the mode of appropriating or dividing the profits, but such alteration being deemed desirable, an Act of Parliament was obtained (4th and 5th Vict., cap. 94), conferring the necessary powers, and RESOLUTIONS were passed accordingly by General Courts to the effect that the appropriation or division of profits should be made every FIFTH instead of every TENTH year; that the share of the Proprietors should be reduced from ONE-THIRD to ONE-FIFTH; that the Reversionary Bonus to be appropriated to Policies thereafter issued, should be in proportion to the amount the Company would have to pay at the death of the person whose life is Assured, and to the number of Annual Premiums paid on the Policy subsequent to the then last appropriation of profits; and that instead of the share belonging to the Proprietors being added to the Subscription Capital, the whole or a part thereof might be paid in cash by way of Bonus, in proportion to the number of Shares held by each Proprietor.

Option was given to existing Policy holders, at the period of alteration, to exchange their Policies for new ones on the revised plan of appropriation of profits, and of this option by far the larger number availed themselves. Of those Policies which were not exchanged, only 23 were in force on the 31st January, 1871. These had the share of

* Commonly called the Assurance Fund.

profits assigned to them which the Deed directs; all the other existing Policies came under the Resolutions.

The table of mortality used for the valuation of whole term single life Policies, with and without participation in profits, was that known as "THE SEVENTEEN OFFICES' EXPERIENCE." The Assurances on joint lives, joint lives and survivors, contingent survivorships, and also the endowment Assurances, were valued by the Carlisle Table. No table was used for the Assurances against issue or the endowment of children, the whole Premium paid being reserved in these cases. The Assurances for terms of years were not valued by any table, three-fourths of the gross Annual Premium being reserved. The Annuities were valued by Mr. A. G. Finlaison's Tables, Report of 1860, Observations No. 14 and 15.

The rate of interest used for the whole term of single life Assurances was 4 per cent., and for those classes valued by the Carlisle Table 3 per cent. The Annuities were valued at 4 per cent.

The portion of Annual Premium reserved as a provision for future profits and expenses, in the case of whole term single life Policies, with and without profits, was the difference between the net 4 per cent. Offices' Experience Premiums and the present tabular rates for the ages at which the Assurances were severally effected. If such tabular rates had always been in use, this method would have been identical with what is technically styled a "Net Premium Valuation," but the rates having been more than once altered, it does not come strictly within that category. The proportion which the amount thus reserved bears to the GROSS Annual Premiums is 28·7 per cent. for Assurances with profits, and 14·9 per cent. for those without profits. The proportion which it bears to the NET Premiums reserved for risk is 40·27 per cent. for Assurances with profits, and 17·48 per cent. for those without profits. The reserve for profits and expenses, in the classes valued by the Carlisle Table, is the difference between the net Carlisle 3 per cent. Premium for the several risks, and the gross Premium charged, but, from the method of valuation adopted for these, the proportion is not ascertained.

The Policies effected in the Java Agency of the Company, at special rates, were first valued as though they had been ordinary Policies at home rates, and to the net reserve thence resulting an addition of 30 per cent. was made.

Policies on the participating scale of premium upon which one premium has been paid, share in the next thereafter ensuing division of profits.

The following is a Specimen Table of Reversionary Bonuses allotted to Policies for £100 under the present quinquennial plan of division, which have been in force for five and ten years respectively:—

AGE AT ENTRY.	DURATION OF POLICIES.	
	5 YEARS.	10 YEARS.
20	£2·50	£5·00
30	2·50	5·00
40	2·50	5·00
50	2·50	5·00

No specimens can be given of Bonuses on Policies of more than ten years' duration, because, being a percentage on sums insured and previous bonuses, the amounts vary for the same period, according as the previous bonuses may have been left attached to the policies, or may have been either wholly or in part surrendered. In actual cases they ranged between £8·750 and £5 per cent. on the sums assured.

The following is a Specimen Table of Bonuses declared upon Policies of the old decennial series for £100. None of these Policies have been in force less than 29 years, or more than 50 years:—

Age at Entry.	Duration of Policies.				
	30 Years.	35 Years.	40 Years.	45 Years.	50 Years.
20	£4·50	£5·25	£6·00	£6·75	£7·50
30	4·50	5·25	6·00	6·75	7·50
40	4·50	5·25	6·00	6·75	7·50
50	4·50	5·25	6·00	6·75	7·50

The only mode of apportionment is by way of reversionary addition. This the assured has the option, at any time afterwards, of surrendering for cash value, or equivalent reduction of premium.

Consolidated Revenue Account for Five Years, commencing the 1st February 1866, and ending the 31st January 1871.

	£	s.	d.
Amount of Funds on 1st February, 1866, the beginning of the Five Years	£958,470	9	3
Premiums (after deduction of Re-assurance Premiums)	421,304	6	6
Consideration for Annuities granted	16,039	11	5
Interest and Dividends	201,210	11	1
Transfer Fees and Discount on Stamps	65	18	10
Balance of Profit and Loss	8,896	13	4
	£1,605,987	10	5

	£	s.	d.
Claims under Policies (after deduction of sums re-assured)	£518,477	8	8
Surrenders and Returns	29,628	13	9
Annuities	1,619	9	4
Commission	15,331	19	3
Expenses of Management	38,536	12	10
Dividends to Shareholders	36,000	0	0
Amount of Funds on 31st January, 1871, the end of the period	966,393	6	7
	£1,605,987	10	5

Summary and Valuation of the Policies of the Imperial Life Insurance Company, as at 31st January 1871.

DESCRIPTION OF TRANSACTIONS.	PARTICULARS OF THE POLICIES FOR VALUATION.					VALUATION. Value arrived at as stated in Answers to Questions 3 and 4, Fifth Schedule.				
	Number of Policies.	Sums Assured and Bonuses.	Office Yearly Premiums.	Office Deferred Yearly Premiums.	Net Yearly Premiums, if ascertained.	Sums Assured and Bonuses.	Office Yearly Premiums.	Office Deferred Yearly Premiums.	Net Yearly Premiums, if computed.	Net Liability.
		£ s. d.	£ s. d.	£ s. d.		£	£	£	£	£
ASSURANCES.										
I. WITH PARTICIPATION IN PROFITS.										
For whole term of Life	2,698	2,102,752 4 5	59,369 2 3	—	—	1,108,548	672,233	—	479,245	629,303
Joint Life and Survivor	2	2,109 1 0	14 10 0	—	—	—	—	—	—	1,159
Joint Life	5	1,000 0 0	43 0 6	—	—	—	—	—	—	36
Extra Premiums payable	—	—	1,993 6 10	—	—	—	—	—	—	—
Total Assurances with profits	2,705	2,105,861 5 5	61,419 19 7	—	—	—	—	—	—	630,498
II. WITHOUT PARTICIPATION IN PROFITS.										
For whole term of Life	630	623,577 5 0	20,543 1 4	2,385 3 3	—	291,934	233,155	28,633	222,825	69,109
Joint Life and Survivor	40	18,620 0 0	339 8 3	—	—	—	—	—	—	4,248
Joint Life	29	10,700 0 0	489 10 8	—	—	—	—	—	—	1,910
Contingent Survivorship	21	31,575 0 0	453 4 11	—	—	—	—	—	—	577
Endowment Assurance	25	8,500 0 0	313 15 5	—	—	—	—	—	—	452
Endowment	15	9,100 0 0	358 1 1	—	—	—	—	—	—	954
Short Period	44	38,125 0 0	924 4 3	—	—	—	—	—	—	741
Issue Cases	6	29,650 0 0	—	—	—	—	—	—	—	615
Sum payable at fixed term	1	1,127 0 0	30 15 2	—	—	—	—	—	—	120
Extra Premiums payable	—	—	1,116 16 6	—	—	—	—	—	—	—
Total Assurance without profits	811	770,974 5 0	24,568 17 7	2,385 3 3	—	—	—	—	—	78,726
Ditto with profits, as above ..	2,705	2,105,861 5 5	61,419 19 7	—	—	—	—	—	—	630,498
Total Assurances	3,516	2,876,835 10 5	85,988 17 2	2,385 3 3	—	—	—	—	—	709,224
Deduct Re-assurances	—	108,350 0 0	4,023 9 11	582 10 5	—	—	—	—	—	3,076
Net amount of Assurances	3,516	2,768,485 10 5	81,965 7 3	1,802 12 10	—	—	—	—	—	706,148
*Adjustments if any	—	—	—	—	—	—	—	—	—	3,261
	3,516	2,768,485 10 5	81,965 7 3	1,802 12 10	—	—	—	—	—	709,409
ANNUITIES.										
Immediate	20	1,743 18 9	—	—	—	14,796	—	—	—	14,796
Other Classes	—	—	—	—	—	—	—	—	—	—
Total of the results			81,965 7 3	1,802 12 10	—	—	—	—	—	724,205

* See Summary of Valuation of Java Risks on next page.

Summary and Valuation of the Java Risks, as at 31st January 1871.

DESCRIPTION OF TRANSACTIONS.	PARTICULARS OF THE POLICIES FOR VALUATION.					VALUATION. Value arrived at as stated in Answers to Questions 3 and 4, Fifth Schedule.				
	Number of Policies.	Sums Assured and Bonuses.	Office Yearly Premiums.	Office Deferred Yearly Premiums.	Net Yearly Premiums, if ascertained.	Sums Assured and Bonuses.	Office Yearly Premiums.	Office Deferred Yearly Premiums.	Net Yearly Premiums, if computed.	Net Liability.
		£ s. d.	£ s. d.			£	£		£	£
ASSURANCES.										
I. With Participation in Profits.										
For whole term of Life	53	62,200 14 0	1,880 11 0	—	—	27,135	25,828	—	18,448	8,687
Other Classes	none									
Extra Premiums payable	—	—	1,085 19 1							
Total Assurances with profits ...	53	62,200 14 0	2,966 10 1	—	—	27,135	25,828	—	18,448	8,687
II. Without Participation in Profits.										
For whole term of Life	29	31,267 0 0	899 17 11							
Short period	4	4,000 0 0	164 5 0	—	—	12,770	12,919	—	10,752	2,018
Extra Premiums payable	—	—	622 7 9	—	—	—	—	—	—	164
Total Assurances without profits ...	33	35,267 0 0	1,686 10 8	—	—	—	—	—	—	2,182
Do. with profits (as above)	53	62,200 14 0	2,966 10 1	—	—	—	—	—	—	8,687
Total Assurances	86	97,467 14 0	4,653 0 9	—	—	—	—	—	—	10,869
No Re-assurances.										
*Adjustment	—	—	—	—	—	—	—	—	—	3,261
Total of the results	86	97,467 14 0	4,653 0 9	—	—	—	—	—	—	14,130

* Being an addition of 30 per cent. to the net liability for Java risks.

Valuation Balance Sheet, as at 31st January 1871.

Dr.

To net Liability under Assurance and Annuity transactions	£724,205	0	0
To Surplus	68,082	0	0
	£792,287	0	0

Cr.

By Life Assurance and Annuity Funds	£792,287	0	0
	£792,287	0	0

The average rate of interest (after deduction of Income Tax) at which the Assurance Fund has been invested in the Five Years from 31st January 1866, to 31st January 1871, is—

1st year	3·752	per cent.
2nd "	3·825	"
3rd "	3·934	"
4th "	4·138	"
5th "	4·193	"

The following are the minimum values allowed for the surrender of Policies of £100 for the whole term of Life, effected at Home Rates, prior to May 1867.

Age at Entry.	When the Policy has been in force					
	5 Years.	10 Years.	15 Years.	20 Years.	25 Years.	30 Years.
20	2·331	5·079	8·303	12.084	16·533	21·524
30	3·476	7·555	12·352	17·733	22·875	29·588
40	5·377	11·409	17·911	24·698	31·570	38·173
50	7·755	15·876	24·088	31·975	39·261	45·779

N.B. The Cash value of existing Bonuses is given in addition.

The above Values are 70 per cent. of those deduced from the "Seventeen Offices' Experience" at 4 per cent. Policies with Profits effected at subsequent dates to May 1867, have 40 per cent. of the Ordinary Premiums received on them, given as Surrender Value, when the Bonuses have not been wholly or in part surrendered. Policies without Profits have 35 per cent. of the Ordinary Premiums received on them, returned on surrender. The Java Policies have one-third of the entire Premium received, allowed on surrender, both with and without Profits, but in the former case the Bonuses must have been left undiminished. Endowments payable at the age of 21 are valued by the Carlisle Table at 3½ per cent. and 80 per cent. of the result is returned on surrender.* The following are specimens for £100 Policies:—

Age.	When the Policy has been in Force	
	5 Years.	10 Years.
1	15·254	32·693
5	19·733	43·615
10	31·830	71·105

* Proof is required however that the health of the endowed child is good at the time.

Endowment Assurances are valued by the Carlisle Table at 3 per cent. and 80 per cent. of the result is returned on surrender. The following are specimens for £100 Policies payable at the age of 60 or at previous death.

Age.	When the Policy has been in force		
	10 Years.	20 Years.	30 Years.
20	11·965	27·224	48·051
30	17·917	42·417	—
40	31·563	—	—

Policies on unhealthy Lives are valued as at the advanced ages charged for.

NORTH BRITISH AND MERCANTILE INSURANCE COMPANY.

Commenced Life Business, 1823.

Extracts from Report by the Directors.

Life Business.

During the five years, from 30th December 1865 to 31st December 1870, the number of Policies issued, the Sums Assured, and the Premiums received, have been as under:—

	No. of Policies.	Sums Assured.	Premiums.
1866	1001	£827,808	£25,161 0 2
1867	916	821,818	29,273 9 10
1868	908	738,582	23,574 0 3
1869	863	933,301	30,922 16 2
1870	879	810,632	24,348 10 0
	4567	£4,132,141	£133,279 16 5

Being an average of 913 in the number of Policies issued, and £826,428 per annum of the Sums Assured.

Quinquennial Investigation.

A copy of the Report of the Actuaries of the Company is appended.

* * * * *

The Directors, in conformity with the Report of the Actuaries, on this footing propose to declare a Bonus of £1 : 5s. per cent per annum on all sums assured in their own Office, and 18s. 6d. per cent on sums insured by the Policies of the United Kingdom Company, being the share to which they are entitled under the agreement with this Company.

Although the Bonus is at the rate of 25s. per cent per annum on the sums assured by the Policies of this Company effected since last investigation, it is much higher on Policies of older standing, and in many instances amounts to £1 : 19s. per cent per annum on the original sum

assured, in consequence of the present Bonus being declared not only on the original sum assured, but also on the previous Bonus additions.

The Directors further recommend that a Prospective Bonus of £1 per cent per annum should be paid on all the Policies of this Company on the Participating Scale existing at 31st December last, which may become claims prior to the next Division of Profits, and 14s. 10d. per cent per annum on the Policies on the Participating Scale of the United Kingdom Company.

The Bonus and Prospective Bonus, as the Proprietors are aware, are not payable on any Policy which has not been five full years in existence.

Report by the Actuaries

On the result of the Valuation of the Company's Liabilities.

The 31st of December 1865 terminated the Sixth Septennial Period of the Company's history as a Life Office, at which date the Directors resolved that in future the financial condition of the affairs of the Life Department should be ascertained quinquennially. It has, therefore, now become our duty, in accordance with that resolution, to report on the Valuation of the Liabilities as at 31st December, 1870, being the close of the first Quinquennium.

* * * * *

The rate of interest adopted in the valuation of the policies was 3½ per cent, with the exceptions after-mentioned, being the rate authorised by the Directors in 1844, and used in all the valuations since that date, and which is nearly equal to the return on the current price of Consols.

The Carlisle Table of Mortality having been used in all the previous investigations, was also that adopted in the present valuation. The special classes of policies were valued on Carlisle Mortality at 3 per cent interest, and the Annuities on the Government [1829] Mortality at 3½ per cent, in conformity with former practice.

The Directors are aware that, in terms of the agreement with the United Kingdom Company, the liabilities under their policies are now liabilities of this Company, and are to be held the same as policies issued by this Company in 1869, except that the division of profits to be allocated to them was to be in the proportion of two-thirds to nine-tenths. In valuing these liabilities, the Carlisle Mortality at 4 per cent was used, reserving a sufficient loading for future expenses and profits. This rate of interest was adopted because, as no new lives have been added since 1861, the policies of that Company are gradually falling in and in the course of exhaustion. This rate also was fixed by the agreement as the rate of interest at which their funds were to be improved, and, as it is a little less than the actual rate of interest received by the Company for the whole of their assets, it is perfectly safe to adopt it.

In the present, as in all previous investigations into the Company's affairs, the loading or addition to the premium to provide expenses, and the formation of future profits, has been carefully reserved; the whole loading thus thrown off has been valued and found to amount to £601,554. Such a provision effectually protects the fund against any encroachment on the natural growth of future profits.

LIABILITIES.

	No. of Policies	Sums Assured. £	s.	d.	Value of Policies. £	s.	d.	£	s.	d.
I. Participating Policies.										
1. Ordinary Policies	9,372	6,270,933	9	0	1,092,310	2	0			
2. Joint Lives....................	128	69,884	0	0	16,110	14	8			
3. Joint Lives and Survivor ...	29	24,223	0	0	4,914	12	9			
4. Endowment Assurances......	336	104,776	0	0	26,850	9	0			
5. Single Payment Policies ...	46	35,114	1	0	25,398	15	2			
6. Terminable Premiums	116	59,350	0	0	13,529	8	4			
7. Quinquennial Scale	37	23,400	0	0	4,354	5	1			
8. Half-Premium Scale.........	320	204,680	0	0	5,883	0	11			
9. Assurance and Annuity Scale	3	800	0	0	63	9	0			
	10,387	£6,793,160	10	0	£1,189,414	16	11			
II. Non-Participating Policies.										
1. Ordinary Policies	2,249	2,330,455	10	0	394,311	14	2			
2. Joint Lives....................	32	166,204	0	0	13,624	3	0			
3. Joint Lives and Survivor ...	49	40,502	0	0	7,459	0	9			
4. Survivorships	64	115,854	0	0	8,519	7	8			
5. Endowment Assurances......	84	45,408	0	0	17,816	8	4			
6. Single Payments	53	146,914	18	1	12,294	16	7			
7. Terminable Premiums	8	4,420	0	0	1,787	12	2			
8. Quinquennial Scale	16	13,050	0	0	1,231	8	10			
9. Period Policies	124	143,408	6	8	1,653	7	4			
10. Endowments	135	29,834	2	0	8,959	1	8			
11. Half-Premium Scale.........	138	237,606	0	0	2,867	1	2			
12. Double Assurance	26	13,255	0	0	2,994	9	0			
	13,365	£10,080,072	6	9	£1,662,933	7	7			
III. Bonus Additions to Policies, amounting to..................	...	561,531	10	7	339,575	14	2			
		£10,641,603	17	4	£2,002,509	1	9			
Deduct Re-Assurances and Value thereof	...	1,420,143	10	0	180,258	14	5			
Total Assurances	13,365	£9,221,460	7	4				£1,822,250	7	4
IV. Contingent Annuities	38	4,740	0	10	4,784	8	8			
V. Immediate Annuities of United Kingdom Company	23	1,089	1	11	7,753	3	8			
Total Annuities	61	£5,829	2	9				12,537	12	4
								£1,834,787	19	8
VI. Unpaid Claims not included in Ledger Accounts	...	...			...			44,363	16	7
Total Liability carried forward		...			...			£1,879,151	16	3

ASSETS.

	£	s.	d.
Amount of Accumulated Funds per State . . .	2,075,193	7	7
Interest current at 31st December 1870, say . .	17,000	0	0
Canada Renewal Premiums due in December 1870, not included in ledger account, say . . .	2,000	0	0
Amount of United Kingdom Policy refunded by order of Court of Session, the value of which is included in the Liability	1,158	0	0
Total Assets	£2,095,351	7	7
Total Liability	1,879,151	16	3
Surplus	£216,199	11	4

It will be seen from the foregoing statement that the surplus or profit which has arisen during the last five years amounts to £216,199 : 11 : 4, exclusive of £10,027 : 1 : 9, which has been paid in Prospective Bonuses.

We believe a large portion of this surplus is due to the comparatively small values which the policies effected within the last ten years bear to the whole, as well as to the effect of the selection of Life.

Recent experience has demonstrated that at the higher ages the Carlisle Table is too favourable to the expectation of life; to meet the additional liability thus arising, and to meet the further claims which may be anticipated in subsequent years from the operation of what is termed by writers on the subject "*Suspended Mortality,*" we would therefore recommend that the sum of £33,925 : 6 : 2 be reserved, and this sum we consider ample for the purpose, and for other contingencies.

The balance, after setting aside one-tenth as belonging to the Shareholders, will enable the Directors to declare a Bonus of £1 : 5s. per cent per annum from the last declaration to North British and Mercantile Policyholders on sums assured and previous bonuses; and, in terms of the agreement with the United Kingdom Company, 18s. 6d. per cent per annum from 1st January 1869 to United Kingdom Policyholders.

Thus the surplus as above	£216,199	11	4
Deducting as a reserve the above sum of . .	33,925	6	2
leaves for division	£182,274	5	2
To which add the amount of Prospective Bonuses paid during the Quinquennium	10,027	1	9
Total realised Profit	£192,301	6	11

Which will be appropriated thus:—

APPROPRIATION.

Amount required to provide a Bonus of £1 : 5s. per cent per annum to North British and Mercantile Policyholders, and 18s. 6d. per cent per annum to United Kingdom Policyholders . .	£163,044	2	6			
Proprietors' Share, one-tenth of £192,301 : 6 : 11	19,230	2	8			
Prospective Bonuses paid . . .	10,027	1	9			
				192,301	6	11

We have also to report that we have valued the whole Annuities granted by the Company, and find the result to be as follows:—

ANNUITY FUND.

Total Amount of Fund per State . . .	£238,583	15	7
Interest to 31st Dec. 1870	713	6	5
Total	£239,297	2	0
Value of 618 Annuity Bonds . .	235,795	13	8
Surplus	£3,501	8	4

GENERAL STATE OF AFFAIRS AS AT 31ST DECEMBER 1870.

LIABILITIES.

				£	s.	d.
CAPITAL,—paid up				250,000	0	0
FIRE DEPARTMENT.						
Fire Reserve Fund	£537,332	8	10			
Fire Premium Suspense Account	185,059	15	8			
				722,392	4	6
				£972,392	4	6
LIFE DEPARTMENT.						
Life Assurance Fund				2,075,193	7	7
ANNUITY DEPARTMENT.						
Annuity Fund				238,583	15	7
				£3,286,169	7	8

ASSETS.

FIRE DEPARTMENT.				£	s.	d.
New 3 per cents £100,000				91,750	0	0
American Government and State Securities				221,580	0	0
Stock of the Company, and Railway Stocks and Debentures				47,800	0	0
Foreign Government Securities				13,022	16	2
Loans on Public Rates, Real Security, and Life Interests purchased				66,559	0	6
Office Premises, Edinburgh, London, and Branches				175,211	18	6
Short Loans in London with Security				213,689	17	1
Balances at Branches and Agencies				114,860	14	8
Sundry Accounts, Edinburgh and London				12,178	3	3
Bills receivable, Cash in Bank and in hand				15,739	14	4
				£972,392	4	6
LIFE DEPARTMENT.						
New 3 per cents £80,000	£74,550	0	0			
Loans on Real Security	1,565,912	8	2			
Loans secured over Public Rates	41,872	9	2			
Loans on Personal Security, with Policies	176,064	3	0			
Loans on Policies within Surrender Values	82,837	3	8			
Carried forward	£1,941,236	4	0	972,392	4	6

	£ s. d.	£ s. d.
Assets brought forward	1,941,236 4 0	972,392 4 6
Post-obits	638 15 11	
Premises Waterloo Place, London	10,688 0 0	
East India Railway Debentures, guaranteed by Council of State for India...	20,150 0 0	
Canadian Govt. and Municipal Securities	41,446 3 1	
Premiums in course of collection, Cash at Branches, Half-credit Premiums, etc.	58,366 1 11	
Government Life Annuities	838 5 2	
Cash in Bank and in hand	1,829 17 6	
		2,075,193 7 7

ANNUITY DEPARTMENT.

Loans on Real Security	£179,132 3 8	
Loans secured over Public Rates	20,426 3 3	
Life Interests purchased	6,897 6 7	
Government Life Annuities	152 18 5	
Teachers' Account, etc......................	352 3 9	
Post-obits and Reversions	31,622 19 11	
		238,583 15 7
		£3,286,169 7 8

The following further particulars are extracted from the Returns under the Fifth and Sixth Schedules of the Life Assurance Companies Act:—

The principles upon which the Valuation is made are:—

First.—The liability of the Company is ascertained by taking the difference between the present value of the sums Assured and the present value of the future Premiums, after deducting the loading.

The rates of interest assumed in the calculation are:—

First—For Policies for the whole term of life, with and without profits, and on which uniform Premiums are payable, the rate of interest is 3½ per cent per annum. On the Policies of the United Kingdom Assurance Company, taken over by this Company, the rate of interest is 4 per cent.

Note.—By the agreement with United Kingdom Company, the funds were to be accumulated at 4 per cent.

Second.—For all other classes of Policies the rate of interest is 3 per cent, except Children's Endowments, on which the rate of interest is 3½ per cent.

Third.—For Immediate Annuities the rate of interest is 3½ per cent.

The proportion of the Annual Premium income reserved as a provision for future expenses and profits is 10 per cent on Non-participating Policies, and 26½ per cent on Participating Policies—except Terminable Premiums and Endowment Assurance Premiums with profits, on which the reserve is 23¾ per cent, and Double Assurance Premiums, whereon the reserve is 7½ per cent. On the Premium income of the United Kingdom Company's Policies, the reserve is

5 per cent on Non-participating Policies, and 15½ on Participating Policies.

Note.—**By agreement with United Kingdom Company, the business was to be managed free of expense, except a charge of 5 per cent for Commissions.**

These principles are determined by the Directors in accordance with the Bye-Laws of the Company.

The principles upon which the distribution of Profits among the Policyholders is made are also fixed by the Bye-Laws of the Company. Policyholders paying the Participation Rates of Premium, participate in the Profits of the Life Department to the extent of nine-tenths, the remaining one-tenth being reserved for the Shareholders. The respective shares of the Profit allocated to the Policies is reckoned according to the sum Assured, with all previous Bonuses added thereto, multiplied by the number of years the Premium has been paid since the last Division. The Directors have also power to declare, at each Quinquennial Division of Profits, a Prospective Bonus on such Policies as may become Claims within the next Quinquennium, but such Bonus must not exceed four-fifths of the rate declared at that Division.

Policies share in the profits from the date of issue, but the Bonus allocated does not vest until the Policy has been five full years in force.

Specimens of Bonuses allotted at 31st December 1870 to Policies for £100 effected at the respective ages of 20, 30, 40, and 50, and having been respectively in force for five years, ten years and upwards, at intervals of five years respectively, together with the amounts apportioned under the various modes in which the Bonus might be received.

Age at Entry.	Duration 5 Years.	Duration 10 Years.	Duration 15 Years.	Duration 20 Years.	Duration 25 Years.	
20	£6 5 0	£6 12 10	£7 1 0	£7 9 8	£7 18 10	Reversionary Bonuses.
	1 14 7	2 0 11	2 7 8	2 16 0	3 5 4	Cash Values thereof.
	0 7 6	0 9 0	0 10 6	0 12 5	0 14 6	Reduction of Premium for 5 years.
	0 1 10	0 2 3	0 2 9	0 3 5	0 4 3	Reduction of Premium for Life.
30	6 5 0	6 12 10	7 1 0	7 9 8	7 18 10	Reversionary Bonuses.
	2 2 3	2 9 8	2 18 0	3 8 2	4 1 8	Cash Values thereof.
	0 9 4	0 11 0	0 12 10	0 15 1	0 18 3	Reduction of Premium for 5 years.
	0 2 5	0 8 1	0 3 9	0 4 10	0 6 6	Reduction of Premium for Life.
40	6 5 0	6 12 10	7 1 0	7 9 8	7 18 10	Reversionary Bonuses.
	2 11 5	3 0 6	3 12 5	4 6 7	5 0 2	Cash Values thereof.
	0 11 5	0 13 4	0 16 3	0 19 10	1 3 6	Reduction of Premium for 5 years.
	0 3 4	0 4 3	0 5 9	0 7 11	0 10 5	Reduction of Premium for Life.
50	6 5 0	6 12 10	7 1 0	7 9 8	7 18 10	Reversionary Bonuses.
	3 4 3	3 16 10	4 9 0	5 3 3	5 19 5	Cash Values thereof.
	0 14 4	0 17 8	1 0 10	1 4 8	1 11 10	Reduction of Premium for 5 years.
	0 5 1	0 7 0	0 9 3	0 12 10	0 18 5	Reduction of Premium for Life.

Summary and Valuation of the Policies as at the 31*st December* 1870 (see pp. 222 and 223).

Valuation Balance-Sheet as at 31*st December* 1870.

Dr.

To Net Liability under Assurance and Annuity Transactions, as per Summary Statement	£2,070,583	13	4
Claims known but not proved	44,363	16	7
To Surplus	219,700	19	8
	£2,334,648	9	7

Cr.

By Life Assurance and Annuity Funds, as per Balance-Sheet	£2,313,777	3	2
Interest accrued but not due	17,713	6	5
Foreign Premiums due in December	2,000	0	0
United Kingdom Policy reponed, and amount of claim refunded by order of the Court of Session—the value of the Policy being included in the Liability	1,158	0	0
	£2,334,648	9	7

Consolidated Revenue Account of the Life Department for Five Years commencing 1*st January* 1866 *and ending* 31*st December* 1870.

Amount of Funds on 1st January 1866, the beginning of the period	£1,682,949	6	0
Premiums after deduction of Re-assurance Premiums	1,335,089	5	0
Interest and Dividends	420,556	5	9
Profit on Investments	2,367	15	5
Recording Fees	289	6	7
	£3,441,251	18	9

Claims under Life Policies after deduction of sums re-assured	£1,087,901	19	0
Surrenders	74,263	5	1
Commission	47,613	5	4
Expenses of Management	125,607	3	8
Bonus to Shareholders, being one-tenth of profits for seven years ending 31st December 1865	15,041	0	9
Purchase of business of United Kingdom and Volunteer and General Companies	8,608	15	11
Annuities, United Kingdom Company	7,023	1	5
Amount of Funds on 31st December 1870, the end of the period	2,075,193	7	7
	£3,441,251	18	9

The average Rate of Interest, after deducting Income-tax, at which the Life Assurance Fund of the Company was invested at the close of each year, during the period since the last investigation, was:—

In 1866	£4	8	2	per cent.
In 1867	4	10	7	per cent.
In 1868	4	8	9	per cent.
In 1869	4	5	11	per cent.
In 1870	4	9	0	per cent.

Summary and Valuation of the Policies as at the 31st December 1870.

DESCRIPTION OF TRANSACTIONS.	PARTICULARS OF THE POLICIES FOR VALUATION.				VALUATION. Value by Carlisle Table at 3, 3¼, 3½, and 4 per cent Interest.			
	Number of Policies.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums, if Ascertained.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums if Computed.	Net Liability.
ASSURANCES.								
I. *Assurances with Profits.*		£	£	£	£	£	£	£
For whole Term of Life	9,372	6,808,104·893	195,983·965	Not ascertained.	3,547,546·315	2,653,338·553	......	1,415,579·995
Joint Lives	128	75,382·063	3,613·512					20,108·044
Last Survivor	29	26,435·479	420·096					5,983·715
Endowment Assurances	336	109,623·305	4,347·723					30,431·759
Single Payments	46	41,860·420	......		Net Liability on these classes ascertained by the valuation of each Policy separately, but the value of Sums Assured and future Premiums in some cases were not ascertained.			30,036·263
Terminable Premiums	116	61,628·864	2,747·379					14,733·360
Quinquennial Ascending Scale	35	24,650·437	572·112					5,406·079
Do. Descending Scale	2	607·917	20·558					131·255
Half-Premium Scale	320	205,576·775	3,734·233					6,507·309
Assurance and Annuity Scale	3	821·875	18·612					72·774
Extra Premiums payable			1,703.804					
Total Assurances with Profits	10,387	7,354,692·028	213,161·994					1,528,990·553
II. *Assurances without Profits.*								
For whole Term of Life	2,249	2,330,455·500	80,000·406	Not ascertained.	1,280,828·272	970,438·097	......	394,311·710
Joint Lives	32	166,204·	8,391·292		Net Liability on these classes ascertained by the valuation of each Policy separately, but the value of Sums Assured and future Premiums in some cases were not ascertained.			13,624·151
Last Survivor	49	40,502·	779·500					7,459·038
Contingent Assurances	64	115,854·	2,138·696					8,519·381
Endowment Assurances	84	45,408·	2,435·219					17,816·416
Single Payments	47	49,414·904	......					7,993·898

Terminable Premiums	8	4,420·	245·575		Net Liability on these classes ascertained by the valuation of each Policy separately, but the value of Sums Assured and future Premiums in some cases were not ascertained.			1,787·609
Quinquennial Ascending Scale	15	11,050·	288·783					675·226
Do. Descending Scale.........	1	2,000·	45·712					556·215
Short Period Policies	124	143,408·333	2,335·631					1,653·367
Endowments	135	29,834·100	1,326·858					8,959·084
Half-Premium Scale	138	237·606·	4,060·527					2,867·058
Double Assurance Policies	26	13,255·	430·219					2,994·451
Against Issue by Single Payment	6	97,500·	......					4,300·930
Extra Premiums payable		......	1,337·700					
Total Assurances Without Profits	2,978	3,286,911·837	103,816·118					473,518·534
TOTAL ASSURANCES	13,365	10,641,603·865	316,978·112					2,002,509·087
Deduct Re-assurances.....................		1,452,015·317	48,420·708					180,258·721
Net Amount of Assurances		9,189,588·548	268·557·404					1,822,250·366
ANNUITIES.								
Immediate	627	27,917·129						242,341·019
Contingent, by Annual Premium	21	4,563·392	1,259·083					3,429·236
Do. by Single Payment.........	8	277·167	......					1,372·721
Deferred	9	330·	74·692					1,190·321
Total of the Results	665	33,087·688	1,333·775					2,070,583·663

INSTITUTE OF ACTUARIES.

The list of candidates who passed the Second Year's Examination is incorrectly given in the Report of the Proceedings of the Institute of Actuaries for the Session 1871–72, which appeared in our last Number (see p. 144). It should stand as follows:—

1. C. D. Higham.
Æq. { G. S. Crisford. W. T. Gray. A. Smither. }
5. G. King.
6. W. Kember.

CORRESPONDENCE.

ON THE PARTIAL COMMUTATION OF PREMIUM.

To the Editor of the Assurance Magazine.

Sir,—The following problem, besides presenting several points of interest, admits also of some useful practical applications. Perhaps you will be able to make room for it in your pages.

Problem.

A person aged x desires to assure his life for the sum A, the Office premium for which is P_x. He proposes to pay only Q (less than P_x), and consents to an equivalent abatement in the sum assured during the next t years. Required X, the amount of the abatement.

There are various cases, according to the form taken by the abatement. I shall consider two.

Case 1. The abatement uniform during the term.

The Office gives up, in premium, $P_x - Q$, the value of which is,

$$\frac{(P_x-Q)N_{x-1}}{D_x};$$

and it takes back, in assurance during the next t years, X, the *net* value of which is,

$$\frac{XM_{x|t}}{D_x}.$$

The commission or loading, say l per unit, on the whole sum nominally assured, is included in P_x. The portion of the assurance taken back by the Office will therefore be allowed for at the same rate. Hence the *Office* value of this portion will be,

$$\frac{(1+l)XM_{x|t}}{D_x}.$$

Equating, we get,

$$X=\frac{(P_x-Q)N_{x-1}}{(1+l)M_{x|t}}.$$

Otherwise:—

The benefit is a whole-life assurance of A, less a temporary assurance of X for t years; and its value is,

$$\frac{(1+l)(\mathrm{AM}_x-\mathrm{XM}_{x|t})}{\mathrm{D}_x}.$$

Also, the premium payable being Q, its value is,

$$\frac{\mathrm{QN}_{x-1}}{\mathrm{D}_x}.$$

Equating,

$$(1+l)(\mathrm{AM}_x-\mathrm{XM}_{x|t})=\mathrm{QN}_{x-1};$$

whence,

$$\mathrm{X}=\frac{(1+l)\mathrm{AM}_x-\mathrm{QN}_{x-1}}{(1+l)\mathrm{M}_{x|t}}.$$

And introducing P_x into this expression, by means of the relation,

$$\mathrm{P}_x=\frac{(1+l)\mathrm{AM}_x}{\mathrm{N}_{x-1}},$$

we have finally,

$$\mathrm{X}=\frac{(\mathrm{P}_x-\mathrm{Q})\mathrm{N}_{x-1}}{(1+l)\mathrm{M}_{x|t}},$$

as before.

It may be noted that X decreases as Q increases, and vanishes if $\mathrm{Q}=\mathrm{P}_x$. If Q exceed P_x we should have X negative, implying that in this case A, instead of undergoing a diminution, would receive an augmentation.

Example. Let $x=30$, A=£1000; then, using the H^{M} Table, at 3 per-cent, with a loading of 20 per-cent, ($l=\cdot20$), we have $\mathrm{P}_x=22\cdot554$. And if Q=21 and $t=10$, the equation becomes,

$$\mathrm{X}=\frac{1\cdot554\mathrm{N}_{29}}{1\cdot20\mathrm{M}_{30|10}}.$$

N_{29}		log 5·887942	page 10.
	1·554	„ 0·191451	
		6·079393	
M_{30}	14521·0		
M_{40}	11870·7		
	2650·3		
$\times\cdot2$	530·1		
	3180·4	„ 3·502482	
X	377·495	„ 2·576911	

That is, the abatement being £377. 10*s*. 0*d*., the sum assured will be £622. 10*s*. 0*d*. during the next ten years, and £1000 during the residue of the life of (x).

The following small table shows the results arising from giving to t the values in the first column in succession:—

t	X	A−X
10	377·495	622·505
20	195·054	804·946
30	128·279	871·721
40	92·811	907·189
50	74·704	925·296
60	69·348	930·652
*ω	68·900	931·100

Case 2. The abatement commencing at X, and decreasing annually by one tth part of X.

The Office here gives up, as before, of premium, P_x-Q, the value of which is,

$$\frac{(P_x-Q)N_{x-1}}{D_x};$$

and it takes back an assurance commencing at X, and decreasing annually by $\frac{X}{t}$ till extinction, the value of which is (*Journal*, vol. xii, p. 343),

$$\frac{X\{M_x-\frac{1}{t}(R_{x+1}-R_{x+t+1})\}}{D_x}.$$

Multiplying by $1+l$ and equating,

$$X=\frac{(P_x-Q)N_{x-1}}{(1+l)\{M_x-\frac{1}{t}(R_{x+1}-R_{x+t+1})\}}.$$

Example. Let $x=30$, A=£1000, $t=10$, all as before. And P_{30} being 22·554, let also Q=21, as before.

The numerator here is the same as in last example; and its logarithm therefore is 6·079393.

The denominator is,

$$1{\cdot}20\{M_{30}-\tfrac{1}{10}(R_{31}-R_{41})\};$$

and it is computed as follows:—

R_{31}	392498·7	
R_{41}	262043·4	
	130455·3	
$\frac{1}{10}$	13045·5	
M_{30}	14521·0	
	1475·5	
×·20	295·1	
	1770·6	log 3·248120
Numerator . . .		„ 6·079393
X	678·067	„ 2·831273

* The limiting value of the table.

Hence the sum assured during the first year is 1000−678·067= 321·933; and the amount for each succeeding year is found by adding 678·067÷10=67·807 to that for the year preceding. The sum assured during the eleventh year is thus £1000; and it remains at this amount during the rest of life.

The deduction at the outset seems here somewhat heavy; but it rapidly diminishes, and vanishes at the end of ten years. Were the term extended to twenty years the deduction at the outset would be only 365·053, and the assurance would consequently commence at 634·947.

I must defer till another opportunity the development of the schemes here shadowed forth. I will now merely mention, that they find their practical applications in cases in which it is arranged that a party who has been "rated up," instead of paying additional premium, shall be subjected to a temporary abatement of assurance.

I am, Sir,
Your most obedient servant,

London, 21 *Oct.* 1872. P. GRAY.

ON THE RELATION BETWEEN THE VALUE OF A POLICY AND THE RATE OF INTEREST.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In the paper on "Extra Premium," by Mr. J. R. Macfadyen, in the current volume of the *Journal*, that gentleman has given, in a footnote on p. 89, a demonstration intended to show that "in any given case it is practically certain that the value of a policy by a higher rate of interest must always be less than by a lower." Having, sometime ago, myself arrived at a similar result to Mr. Macfadyen's by a rather different process, I venture to send it you, with the hope that it may be of interest to some of your readers.

We have, by a well-known formula,

$$ {}_nV_x=1-(1-V_x)(1-V_{x+1})\ldots\ldots(1-V_{x+n-1}); $$

consequently, it will be sufficient to consider how the value of a policy one year old is affected by increasing or diminishing the rate of interest at which it is calculated.

Now,
$$ V_x=1-\frac{1+a_{x+1}}{1+a_x} $$

$$ =1-\frac{a_x}{vp_x(1+a_x)} $$

or, omitting the subscript x,

$$ V=1-\frac{a}{vp(1+a)}. $$

Differentiating this with respect to v, we have

$$\frac{dV}{dv}=-\frac{1}{p}\cdot\frac{v(1+a)\frac{da}{dv}-a(1+a)-av\frac{da}{dv}}{v^2(1+a)^2}$$

$$=-\frac{1}{p}\cdot\frac{v\frac{da}{dv}-a(1+a)}{v^2(1+a)^2}.$$

Now, $a=vp+v^2{}_2p+v^3{}_3p+\ldots\ldots$

where ${}_np$=probability of x living n years.

$\therefore$ $$v\frac{da}{dv}=vp+2v^2{}_2p+3v^3{}_3p+\ldots\ldots$$

$$=a_x+vpa_{x+1}+v^2{}_2pa_{x+2}+\ldots\ldots$$

Also, $a(1+a)=a_x+vpa_x+v^2{}_2pa_x+\ldots\ldots$

Hence, if a_x is greater than a_{x+1}, a_{x+2},......*i.e.*, if the value of an annuity on x's life is greater than the value of an annuity on any life older than x,

then $$a(1+a)>v\frac{da}{dv};$$

that is, $\frac{dV}{dv}$ is a positive quantity: in other words, as v increases so also does V. But as $v\left(=\frac{1}{1+i}\right)$ increases, i diminishes; therefore the lower the rate of interest the greater is the value of a policy of one year's standing on x's life.

Similarly, if a_{x+1} is greater than a_{x+2}, a_{x+3}......, V_{x+1} increases in value as the rate of interest diminishes; and generally, if the values of annuities on the lives x, $x+1$, $x+2$,...... form a continually decreasing series, the value of a policy of one year's standing taken out at any age from x upwards is greater the less the rate of interest. Whence it follows from (1) that ${}_nV_x$ increases as the rate of interest decreases.

I am, Sir,
Your obedient servant,
W. SUTTON.

18 *Lincoln's Inn Fields*,
26 *August* 1872.

ERRATUM.

The last line on p. 4 of this volume is misplaced and should be carried over so as to be the last line on p. 8.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

On Reversionary Life Interests as Securities for Loans. By T. B. SPRAGUE, M.A., *Vice-President of the Institute of Actuaries.*

[Read before the Institute, 25 November 1872.]

OF all the securities proposed to insurance companies, reversionary life interests are the most troublesome to deal with in the ordinary way of mortgage; and the objections to so dealing with them have been considered by many actuaries so serious, that they have laid down the rule that the only safe way of making an advance on the security of a reversionary life interest is by way of reversionary charge. The transaction, they think, should be in the nature of a sale rather than a mortgage, a portion of the reversionary life interest being sold in consideration of the present advance.

When a borrower applies for a loan on a reversionary life interest, he generally expects to be called on to effect an insurance of about the same amount as would be required to secure a loan on an immediate life interest; and it is difficult to make him understand why a greatly larger insurance is necessary to protect the lender. The reason becomes obvious enough when we consider what remedy a lender has in the event of the borrower failing to pay his interest and premiums. In that case, there appear to be three possible courses open to the lender. First, he may, by agreement with the borrower, allow the premiums and

interest to accumulate at compound interest; or, secondly, he may sell the reversionary life interest; or, thirdly, foreclose. Suppose the amount of the insurance that has been effected to be such as would be required in the case of an advance on the security of a life interest in possession, or to exceed the sum lent by about 20 per-cent. Then, if the first of the above courses is adopted, and the interest and premiums are allowed to accumulate at compound interest, it is clear that in a very few years, (probably within three years, under the circumstances supposed) the accumulated amount of the loan will exceed the insurance, and a further insurance will become necessary to protect the lender from loss by the death of the borrower. Should the borrower be still in good health, no difficulty arises. If, however, he has fallen into bad health, and his life is only insurable at a greatly increased premium, the value of his reversionary life interest will be much reduced; and unless the sum originally lent was but a small fraction of the value, the lender may probably find that the security is worth less than he has lent upon it. But it may happen that the borrower's life has become wholly uninsurable. If it were then certain that he would die within a short term of years, the difficulty would be met by increasing to a moderate extent the original amount of the insurance; but, as is well known to all persons familiar with life insurance business, a person whose life is practically uninsurable may nevertheless live for ten or even twenty years; and if the borrower's life in the case supposed should be extended in this way, the accumulations at compound interest would make his debt very greatly exceed the amount of the insurance, so that the lender would receive on the death of the borrower only a small part of the sum due to him. It appears then that, in this case, in order to protect the lender completely, the full insurance that will be ultimately required should be effected (or arranged for) at the outset when the loan is originally granted. Secondly, suppose that the lender attempts to sell the reversionary life interest; then since the purchaser will in no case receive any income until the death of the life tenant, and will receive nothing at all if the reversioner should die before the life tenant, he will of course require to have an insurance on the reversioner's life sufficient to cover the probable amount of the accumulations at compound interest of his purchase money and the premiums which he may pay before he comes into possession of the reversionary life interest. Here the same difficulty meets us; for, under the circumstances supposed, a new insurance will have to be effected when

the reversionary interest is put up for sale, and it may happen that the reversioner's life has become uninsurable; in which case the reversionary life interest will be perfectly unsaleable. Lastly, if the lender forecloses, he practically becomes the purchaser himself, and the same remarks apply.

From these considerations we conclude that, in order to make a loan in the ordinary way of mortgage on the security of a reversionary life interest, an insurance on the reversioner's life must be either effected or arranged for at the outset, of sufficient amount to render the reversionary life interest practically saleable, at such a price as to return the lender the amount of his advance with all arrears of interest and premiums and legal costs. Suppose, for example, that the life tenant is 60 and the reversioner 30, then a reversionary annuity of £1000 will be worth £4276, and the policy necessary to protect a purchaser fully will be £14,286 (see the tables appended to my paper "On the Valuation of Reversionary Life Interests" vol. xiv, pp. 432, 3). The annual premium on this policy will be £319. 13*s.*; and the very largest sum that could be lent on the security of the annuity would be £3500. In this case, then, the amount of the insurance exceeds four times that of the advance, even when we take the smallest margin consistent with safety. Taking the interest at 5 per-cent, the annual sum which the borrower is required to pay is £495, or more than 14 per-cent on the advance. If the loan is to be allowed to accumulate at compound interest, of course the amount that could be advanced is greatly reduced. Thus, for example, if in the above case the interest and premiums are to accumulate at 5 per-cent compound interest for 5 years, the sum to be lent must be so fixed that the accumulated amount of the debt at the end of the five years shall be so much less than the then value of the reversionary annuity that there is no doubt, in the event of that annuity being sold, the lender will receive the full amount of the debt and costs. The value of the reversionary annuity of £1000 and the policy of £14,286, at the end of the five years will be very nearly the same as that of a similar annuity on a life of 30 expectant on the death of 65, which the table referred to above gives as £5227. The accumulated amount of the debt at the end of five years should therefore not be more than about £4700; whence it follows that the original loan should not be more than £2229, so that the insurance is no less than 6·4 times the sum lent.

In all the cases we have considered, it is to be observed that the large policy is not required at the outset of the transaction, but is

only necessary for the purpose of guarding against the risk of the borrower's life becoming uninsurable. If the loan is to accumulate at compound interest, it appears at first sight as if an increasing insurance, which should always exceed the amount of the accumulations by a fixed sum, would meet the requirements of the case, as being sufficient to protect the lender and calling for the smallest possible payment from the borrower. But on closer examination this will, I believe, be found to be impracticable. In order to meet the borrower's convenience, the term of the accumulation would have to extend until the death of the life tenant; but unless the latter were of a very advanced age, probably no insurance company could be persuaded for any reasonable consideration to grant an insurance increasing yearly until his death. If the term of accumulation and the term for which the policy increases is to be a moderate term of years only, some new arrangement will have to be entered into at the expiration of that term; and whatever the nature of that arrangement may be, a further insurance would be required beyond the insurance already arranged for, and we are thus again met by the risk of the life having become uninsurable. If instead of effecting at once the full insurance that is necessary to protect fully the security, an increasing insurance is effected which increases by annual steps thro' a course of, say, 20 years, till it reaches the full amount, the cost to the borrower will be somewhat reduced. If, in this case, the premium on the increasing insurance is a uniform one, the saving to the borrower will, I believe, be comparatively unimportant. This, however, is a point which appears to require further investigation. On the other hand, insurance companies would be reluctant to grant such an increasing insurance at a premium increasing proportionately. The immediate cost to the borrower may be reduced by effecting the insurance on the ascending scale of premium or on the half-premium plan; but this involves a still greater increase of his ultimate payments, if the reversion does not fall in early.

The foregoing arguments appear to prove beyond all question that in general it will be much more satisfactory, both for borrower and lender, that the advance should be made by way of reversionary charge rather than by way of mortgage. Altho' the necessity of a large insurance is amply demonstrated, the terms of the mortgage appear so onerous to the borrower that the transaction is likely to bring a lender into disrepute. Generally speaking, the borrower is not in a position to pay the interest on the loan and the premiums on the heavy insurance necessary to secure the advance properly;

and the only way in which he can practically find the means to do so is by borrowing further sums from year to year as his payments grow due, by which means his indebtedness accumulates at a truly frightful rate. If, however, the advance is made by way of reversionary charge, he is called on for no payment until he comes into possession, and he knows exactly what payments he will then have to make out of the income he receives. On the other hand, the lender will receive nothing until the death of the life tenant. This circumstance renders the transaction almost always an unsuitable one for an individual lender, but forms no objection when the lender is a life insurance company; in fact, almost all life insurance companies, and more particularly the young and growing companies, have constantly sums coming into their coffers from premiums, interest, and repayment of loans, for which they are seeking investments; and it is only in the case of a very old company, in which the outgo largely exceeds the income, and the invested funds have to be realized in order to pay the claims, that transactions of this nature are quite unsuitable.

I shall, therefore, throughout the remainder of this paper, consider that the lender is a life insurance company; and I take this opportunity of repeating what I have said on a former occasion, that I think transactions of this nature are extremely well suited for the investment of the funds of well established and prosperous life insurance companies. Altho' an office making large advances by way of reversionary charge runs the risk in every case of sustaining a loss on the investment, through the life tenant and the reversioner jointly living beyond their expectation, yet it is fully compensated for this risk by the chance it has of making a profit by the early death of the life tenant in other similar transactions, and the increase of its insurance business caused by the large insurances they introduced.

Assuming, then, that the advance is made by way of reversionary charge, it is to be observed that the arrangement admits of various modifications, which I propose to consider briefly. It may firstly be stipulated that the borrower shall pay an annuity, to run from the day of the death of the life tenant until that of his own death; and in this case it is usual to name beforehand a sum for which the annuity can be redeemed, at the option of the borrower, at any time after it has become payable. It may, secondly, be stipulated that the borrower shall, upon coming into possession, pay a fixed sum to the office; and in this case it is necessary to add that, until he pays such sum, he shall pay interest

upon it at a fixed rate and also the premiums on policies sufficient to secure the charge. Each of these methods has its own advantages for the company. In both, the company must, at the time of making the advance, effect policies for the whole term of the borrower's life of such an amount as is considered sufficient to secure the transaction. In the latter case, where the office is entitled to demand payment of the charge on the death of the life tenant, and of course has the usual power of sale, which it can exercise whatever may be the state of health of the borrower, justice requires that the policies shall become the property of the borrower, from the date when he becomes liable to pay the charge. They will then be treated as mortgaged to the company to secure the amount of the charge and interest due to it; and any excess of insurance beyond that amount will belong to the borrower's estate. It seems open to question whether the policies should not be considered in this light from the very outset. The office will naturally effect policies without profits, because the cost of such is less than that of participating policies. In the former case, it will be more advantageous to the company to effect policies with profits, for the reason that in fairly successful offices, the bonuses declared on a policy greatly exceed in value the difference between the participating and non-participating premiums. Then, if the borrower does not exercise his option to pay off the charge, the company has the benefit of the bonuses accruing from time to time on the policies, which may easily, if the offices have been well selected, increase the rate of interest realized by one per-cent per annum. If the borrower redeems the annuity, his right to the policies is matter of arrangement, and it seems fair that the bare policies should be assigned to him, the bonuses being retained by the company. It is clearly unjust that the policies should belong absolutely to the company; but as the company is under no obligation to effect participating policies, it is reasonable that if it pays the higher premium required for them, the bonuses thus gained should be, in any event, its property. When the terms of the advance are arranged in the latter of the above methods, the company, in addition to the right of requiring payment of the charge, has the advantage of being subject to a smaller deduction for income tax; for this is calculated only on the interest, whereas, in the case where the company receives an annuity, the tax is calculated, not on the portion of it which represents the interest on the cost, but on the full amount. As I mentioned in my paper above referred to, when the advance is made in the former of the above

methods, it seems the best course in practice for the company to effect at the outset policies considerably less than the amount required fully to protect the security, trusting that the bonus additions to the policies will increase the sum assured to a sufficient extent by the time the annuity becomes payable.

In the case where the amount borrowed is very small in comparison with the value of the reversionary life interest, so that the reversionary charge is either less or but little more than a year's income of the property charged, it is sometimes thought sufficient, instead of effecting a policy for the whole term of the reversioner's life, to effect one on his life against that of the tenant for life and for, say, 3 years longer. In this case it is of course stipulated that the borrower shall pay a fixed sum on coming into possession; and it is anticipated that if he should fail to do so, or to make some other satisfactory arrangement, within, say, a year after the death of the life tenant, the company would have no difficulty in going into receipt of the rents or other income, and obtaining the full amount of the charge with interest, premiums, and costs, before the expiration of the insurance.

The formula for the amount of the reversionary charge in this case is precisely the same as that by which the value of a contingent reversion is found. The value of 1 to be paid by the borrower aged x, on the death of the life tenant aged y, if he is then alive, is

$$1-(\mathrm{P}+d)(1+a_{xy})\,*;$$

P, in the case under consideration, denoting the annual premium for an insurance on the life of x against y and for 3 years longer. This assumes that the amount of the insurance effected is the same as that of the reversionary charge. It is true that in practice the reversionary charge will bear interest from the day of the death of the tenant for life, and that the lender may also have to pay the premium on the policy, and law costs; but as by supposition the income of the mortgaged property is large in comparison with the loan, and if the borrower should die there will be always current income available towards paying the charge, it seems unnecessary to require a larger insurance than for the amount of the loan. It is of course provided, that until the borrower pays the amount of the charge and interest, he shall pay the premiums that fall due on the policy, those which fall due before the death of the life tenant being paid

* If a single premium is payable for whole world license or for an insurance against the birth of issue, this must of course be subtracted from the above formula.

by the lender. If now the policy is drawn in the ordinary way, it may happen that the life tenant dies the day after the lender has paid a premium, so that the borrower would have no premium to pay for a year; and, on the other hand, the life tenant might die the day before the premium fell due, in which case the borrower would immediately have to pay a premium. To avoid this inequality, it seems better to provide that the premiums falling due before the life tenant dies shall be commuted by a single payment, and that the first premium to be paid by the borrower shall fall at a fixed time, say 3 or 6 months after the death of the life tenant. Taking the interval as 6 months, the policy would not be on the life of x against y and 3 years longer, but on the life of x against y and $2\frac{1}{2}$ or $3\frac{1}{2}$ years longer. It seems, in fact, only fair to the borrower to give him a fixed time, from the day of his coming into possession, in which he can pay the charge and interest without being called on for any additional payment on account of premiums. Assuming the reversioner to have come into possession, he may prefer, if in good health, instead of paying the charge, to borrow the amount on mortgage of his life interest in possession, effecting a policy for the whole term of his life; or he may prefer to pay the charge out of income. In the event of his choosing the latter alternative, it is only fair that as his debt is reduced, the insurance to secure it should be also reduced; but this is a matter of detail which it would be quite unnecessary to insert in the deed.

Another method which has been suggested for the purpose of reducing the cost of the transaction to the borrower is to effect a policy on his life, subject to a low premium during the life of the life tenant, and a larger premium afterwards. It is clear that the premium to be paid during the joint lives must be not less than would be required for the insurance of x against y; and assuming arbitrarily a premium somewhat larger than this to be payable during the joint lives, it is a simple matter to calculate what premium should be payable after the death of y in order that the present value of the total premiums may be equal to the ordinary whole life premium on x. But in practice this arrangement is not applicable; for under it the borrower has an option for which the company has no equivalent. If the life tenant should die soon, x, if in good health, would be able to effect an insurance for the whole term of his life at a rate considerably lower than the agreed rate, and there would be nothing to prevent him going to another company and borrowing from them in the

ordinary way on his life interest in possession a sufficient sum to pay off the company. On the contrary, if y should live to a great age, the company would be bound to continue the policy on the life of x at the agreed rate. Thus, then, if the contract turns out to the disadvantage of the borrower, he would be able to repudiate the agreed terms, while if it turns out to the disadvantage of the company, they would be bound to them. It is therefore necessary to arrange the terms somewhat differently, and the simplest course appears to be to agree that the policy may be continued in force after the death of y by payment of the ordinary premium for x's then age. In order to compensate the office for the risk of x being then in bad health, the premium payable during the joint lives should be greater than the premium for a contingent insurance, and might perhaps be taken equal to the premium payable during the joint lives for an insurance of x against y and for 3 years longer. Supposing this arrangement resorted to, the company still runs the risk that the joint lives of x and y may be so long extended that the premium to insure x's life on the death of y would be very large; and the value of his life interest consequently diminished. It is therefore clear that this arrangement could only be resorted to with advantage when the value of the security is very ample, and the life tenant is of advanced age.

Altho', as we have seen, it is generally the best both for borrower and lender that advances of the kind we are considering should be made by way of reversionary charge, there are sometimes circumstances which render it undesirable. The tenant for life may be in a bad state of health, or what comes to the same thing, the reversioner may believe, if he does not hope him to be so; and in this case, the reversioner will be extremely unwilling to enter into a contract by which, if his anticipations of the tenant for life's early death should be realized, he would be a heavy loser. A second case which may happen is that the reversioner may have an income sufficient to pay premiums and interest without being in a position to charge that income. This may be the case if he, for instance, has a large allowance from his father, or has married a wife with a large income settled on herself, but paid quarterly to his bankers. Another instance which has recently come under my notice, was one where the borrower had a considerable present life interest in foreign securities, which insurance companies do not consider suitable security for a loan, but he had also a reversionary life interest in real property in England, which is considered an unexceptionable security. In such cases as these it is no longer

true that it is best both for borrower and lender that the advance should be made by way of reversionary charge. Probably in almost every case where it is proposed to raise money on the security of a reversionary life interest, the borrower in the first instance protests more or less strongly against an arrangement by which he will be a heavy loser if the life tenant should happen to die early, overlooking the fact that if the advance is made by way of mortgage, and the life tenant should live to an advanced age, he would have to pay ultimately a much larger sum than under the other arrangement. In general, the actuary should disregard these protests, and insist on the advance being made in what he believes to be the best method for all parties; but he should not lay down one inflexible rule. He should rather be prepared to consider each case on its own merits, and to admit of a departure from his general rule when special circumstances call for it.

The skilful actuary will further not content himself with considering every possible way by which he can protect the interests of his company without any regard to the interests of the borrower, as almost appears to be the policy adopted in some instances; but he will, in the first instance, endeavour to ascertain the lowest possible terms on which the transaction can be carried out with advantage to the company. Having ascertained these, it by no means follows that they will be the terms quoted to the borrower, but he will know that everything beyond is clear profit; whereas, if he has simply proceeded on the plan of making the office safe, he will not in the result know what is the real profit on the transaction, and it may happen that the terms quoted by him when examined by an impartial actuary are found so exorbitant as to bring his company into disrepute. The actuary may in fact consider himself to be an adviser to a certain extent in the interest of the borrower; for it will be his aim to quote the lowest terms consistent with a fair remuneration to the company. Applying these principles, now, to the above mentioned cases, it must be admitted that if there is a moral certainty of the borrower being able punctually to pay interest and premiums, he may fairly require the loan to be made by way of mortgage. He may furthermore very fairly object to being charged at once with the premium on the large insurance which we have seen is necessary to make the security saleable in case of his default. That large insurance only becomes necessary when the security has to be sold, nor will it assist matters at all for the insurance to be an increasing one, for so long as the borrower keeps down the payments required of him,

the necessity for an increase of the insurance will not arise, and he may as properly object to pay the premium on the increased insurance as he objected in the first instance to pay a premium on the large insurance necessary fully to protect the security.

We see, then, that the lender requires no increase in the assurance, so long as the borrower pays his interest and premiums regularly. If the borrower should make default and it should become necessary to sell the reversionary life interest, a larger insurance will certainly be required, but in that case we have seen that the larger amount of insurance is not immediately required by the purchaser, and that it would sufficiently well answer his purpose to have an increasing insurance. What is wanted, therefore, is an insurance that shall be capable of increase at the option of the assignee of the policy, subject to such conditions as may be agreed on. This, so far as I know, is an entirely novel mode of carrying out the transaction, but it may interest the members of the Institute to know that it is not a mere theoretical proposal, but that under my advice a considerable loan proposal has been recently carried out by means of it.

It would be objectionable to give the option of increase to the borrower himself, because he would be almost certain to exercise it to the disadvantage of the insuring company in the event of his falling into bad health. It may, of course, still happen, when the option is only given to an assignee, that a collusive assignment may be made by the borrower for the mere purpose of exercising the option, but the risk of this is not so great. The risk of loss to the company through this cause is also greatly reduced by the circumstance of the increase in the sum assured taking place, not all at once, but by stages extending over a series of years, which arrangement, as we have seen, will be a sufficient protection to the purchaser. In the case I have already alluded to, a present insurance of £20,000 was effected by the borrower, which was subject to increase up to £40,000. Two policies for £10,000 each were issued, each with an independent option attaching to it, by which means an additional facility would be given to the purchaser of the life interest without any detriment to the company, and on each of the £10,000 policies was placed an endorsement to the following effect:—

In consideration of the within mentioned annual premium, it is hereby agreed that the X Company will, on the application of any *bonâ fide* assignee of the within written policy at any time during the lifetime of A. B., the father of the within mentioned

assured, and during the continuance of this assurance, grant, without further evidence of health, an increasing assurance on the life of the said assured, commencing at the sum of £1000 and increasing at the rate of £1000 a year up to £10,000, the premium on each £1000 of the said assurance being calculated according to the published rate of the said company for an age five years older than that of the said assured at the date of such application, provided that the said company shall not be bound to renew the said increasing assurance from year to year, unless the within written assurance is also continued in force.

For the protection of the offices granting this insurance, more especially considering its novel character, it was thought desirable to limit the option as far as is consistent with its being effectual for the purpose for which it is required, and for this reason it was stipulated that the right of exercising the option shall only continue during the lifetime of the tenant for life. On his death the borrower comes at once into possession of the income, and if it has not been necessary to sell his interest during the lifetime of the tenant for life, the original policy will be amply sufficient to secure the advance after the death of the tenant for life.

When a loan proposal is carried out in this way, it must not be overlooked that every year which passes without the option being exercised, increases the premium that would have to be paid on the new insurance, and consequently diminishes the value of the life interest supposed to be in possession. It is clear, therefore, that this could not be entered into as a permanent arrangement, except in cases where the margin is ample, and the life tenant is advanced in life.

It remains to consider on what terms such a policy might be practically granted by an office. It is clear that as an option is given to the holder of the policy, which option may possibly be exercised greatly to the detriment of the office, it is right that the office should receive a fair equivalent for this option. At present there seems no means of calculating the money value of this option with anything like scientific accuracy, but in the particular case to which I have already referred, it was considered sufficient to charge an additional 5*s*. per-cent per annum on the amount of the original policy, and on the above terms several first-class offices agreed to share the risk.

The principles here adopted may without difficulty be extended to the case of advances made on contingent reversions, to which many of the preceding remarks apply with very little alteration.

The following account of the discussion which followed the reading of the paper is abridged from the *Insurance Record.*

Mr. A. H. Bailey—The council having referred Mr. Sprague's paper to me, I at first thought I had an easy duty to perform—viz., to decide whether it was a suitable paper to be read in this room. I am sure the meeting will be unanimous in their opinion that it is very suitable for discussion by this Institute. But Mr. Sprague has informed me that the duty of referee extends beyond this—that he is expected also to express his opinion upon the paper submitted to him by the council, which adds considerably to the burden of the referee's duty. In turning the matter over in my mind, I have felt that there is a peculiar difficulty in discussing this subject here. Most of the gentlemen in this room are interested in these transactions on behalf of the lenders; and if we could hear counsel for the borrowers, perhaps additional light would be thrown upon the subject. While I agree with Mr. Sprague that transactions of this nature are well suited for the investment of the funds of an assurance company, there are, notwithstanding, some difficulties in the way. One arises from the apparent, perhaps, rather than real severity of the terms on which these transactions are effected. It would be very difficult indeed to persuade any borrower that if, for every £1 he borrows, he has to pay, say, £4, he is not hardly used. The result is that an assurance company, rightly or wrongly, acquires a certain amount of repute for hard dealing. But there is a more serious difficulty, and that is how to deal with these transactions as a matter of book-keeping. If a considerable sum is advanced, a large amount has to be disbursed annually for life assurance premiums, and, as we all know, the value of reversions, looked at in the ordinary way, increases so slowly that it would hardly do not to credit the account with interest; otherwise, if there are many of these transactions, the average rate of interest on the whole funds would apparently be reduced. I believe it is the practice of some offices to credit year by year 5 per-cent on their outlay. If they do that, the account in the ledger, in the words of the paper, "accumulates at a truly frightful rate," and the amount apparently advanced on these transactions is considerably in excess of the value by any ordinary estimate. This is an inconvenience for which I have never yet heard any practicable remedy suggested. Still, notwithstanding these difficulties, I am bound to say that I think the balance of advantage is in favour of the assurance companies, and that these are very desirable transactions for them. I cannot think they are equally desirable for the borrower.

The case which Mr. Sprague has instanced at the end of his paper, is an interesting, but an extremely rare case. As a rule, reversionary annuitants are in possession of but small immediate means, and it is very seldom indeed that there is any hope that the premiums and interest can be kept down by them. In former times attempts were made to carry these transactions out by way of mortgage, but they almost invariably broke down and proved unsatisfactory. Sometimes efforts were made to give sureties for the premiums and interest, but solvent sureties are not easily found if the borrower is not in a position to keep up the payments himself; and it

may be laid down as a cardinal rule in all investments, that it is unwise for an assurance company to enter into transactions with a borrower which there is no reasonable prospect of the borrower himself being able to carry out. I, like many others, have turned over in my mind whether any way could be devised by which loans could be granted on securities of this description on apparently less onerous terms to the borrower. There are two classes of reversionary interests,—one in which the borrower is tenant in tail; the other in which he is tenant for life in reversion. In many of these cases he is tenant in tail;—because, as is well known, by the law of this country land can be settled only for twenty-one years beyond the life of any person in existence at the period of settlement;—and if he be tenant in tail, although he cannot grant a charge on the fee simple in reversion, if he succeeds to the property he then will be in a position to disentail; and before he succeeds, he can covenant to disentail. In this case it appears to me that the grant of a reversionary charge with an assurance effected on the life of the borrower against that of the tenant for life and for a short term longer, would fairly well meet the case. In the other case, where the borrower is only tenant for life, it has occurred to me that some such arrangement as this might be made. The advance might be made by way of reversionary charge, computed, not by the formula—

$$1-(\mathrm{P}+d)(1+a_{xy}),$$

which Mr. Jellicoe first suggested, and which is one I have never used for myself, for the reason that in all these reversionary transactions, as far as I know, the lenders never do purchase these annuities, —but by the old-fashioned one

$$\mathrm{A}_{xy}-\mathrm{A}'^{1}_{xy},$$

where A_{xy} is the value of the reversion at the death of the joint lives at the current rate of interest, and A'^{1}_{xy} the single premium for the contingent assurance; and then the borrower should endeavour to induce the assurance companies—(and I really see no reason why they should object)—to grant an assurance on the life of x against y by a single premium, and undertake to grant a whole term assurance on the life of x when y dies, without reference to the then state of health of x. Of course a higher premium than the ordinary one should be charged, but I see no reason why they should not enter into a transaction o that sort.

Before I sit down, perhaps I may be permitted to express my opinion of the value of papers of this kind to students and younger members of the Institute. There is no difficulty in learning from any of the text-books the theoretical investigations required to determine the value of reversionary life interests, but to deal with these interests in practice is matter of considerable difficulty. I have for one been a little surprised, as an examiner of the Institute, at finding that, whereas the candidates come up in large numbers for the first year's examination, and in respectable numbers for the second year's, they come up in very small numbers indeed for the third year's examination, which, it appears to me, is by no means the most difficult of the three. The only reason I can find for this state of things is that the information re-

quired for the third year's examination, which is very necessary for the practice of an actuary, is not to be acquired by books; and therefore I conceive that the opportunity which Mr. Sprague has given us to-night, as well as on previous occasions, of getting at some of the knowledge which floats about and is current in assurance offices, is of great value to the junior as well as the senior members of the Institute.

Mr. MACFADYEN—The first part of the paper shows the onerous nature of the terms requisite to protect the mortgagee of a reversionary annuity,—the second indicates various modes in which the *purchase* of such annuities can be arranged,—and the third treats of the manner in which a borrower, if still obstinate enough to mortgage rather than to sell, may have the terms made, if not easy, at least as easy as they can well be. Examining, as the other parts are connected, the second section first, I find that Mr. Sprague, in commenting on the final disposal of the policies of assurance, though willing to hand them over to the seller on the payment of the charge, considers the purchaser should retain the bonuses. Looking at this in connection with the statement that profit policies for less amount than will be ultimately required should be taken out, it seems to me scarcely fair to the seller if he has had deducted by the purchaser in the price given these profit premiums, to retain from him on settlement that part of the bonus equivalent to the difference between the full amount required at the non-profit rate and the sum actually assured. In short, whether it be bonus or otherwise, the seller, on redemption, is entitled to an insurance policy equal in amount to that which the premiums he paid would purchase by the non-profit rates of the office. If, beyond this, the buyer chooses to enter into a speculation by paying an additional premium for profits, of course, as the seller has not been charged this addition, he has no claim to the results. With reference to the first part of the paper, since it is requisite that the full ultimate assurance amount be secured from the first, it is obvious that the borrower is called on to pay a sum calculated on the assumption that he omits to pay premiums and interest from the commencement; and this, even though he make no such default whatever, or at most, for only part of the time. So it is clearly the borrower's interest to be what he is credited with being, and sell his reversion rather than retain it at such a cost. On the other hand, as is proved in the paper, the lender can demand no less. In fact, there is an *à priori* probability in favour of default and uninsurableness going together, as the one may be a direct consequence of the other. Since then, as in the credit system in ordinary business, the borrower must pay a higher price, however punctual he may be in his payments, simply because he might have been a defaulter, all that can be done for him is to make this extra payment as light as possible. Mr. Sprague considers the objections to doing this by an ordinary increasing assurance insuperable, but thinks it might be done by an optional increasing assurance. Taking the actual instance given in the paper, and comparing an insurance of £20,000 increasing to £40,000 with an insurance of £20,000 with an option to increase to £40,000, of the two risks, since there are no means of valuing this option, the former seems to me much the preferable to the society.

No doubt the latter, if it can be obtained at a less rate, will be cheaper for the borrower. On the other hand, it is clear that in the former case the society gives value for the premiums received in the shape of a life insurance, and on the settlement of the transaction this assurance will become the property of the borrower. Of course, I am here assuming that the premiums and interest are paid regularly by the borrower, as if not, he has no grievance in the amount he has to pay. He may not want this large assurance any more than the gentleman, borrowing from certain money-lenders, wants the pictures given him as part of his loan, but at least, unlike the pictures, it is not sold him at a fancy price, but its cost is the result of mathematical calculation. In the latter case, unless by instinct the right value of the option has been hit upon, somebody is aggrieved. Either the office has not sufficient remuneration for the risk undertaken, or the borrower has his pictures sold him at a fancy price. Indeed, if the option principle is to be introduced at all, it seems to me more thorough, if a dealer in such transactions is making the advance, to directly assume that a certain number of borrowers will become defaulters and uninsurable, and calculate accordingly. It may be urged that there are no data for fixing what this number will be. Very true, but there are just as many as for fixing the premium for an optional increasing assurance.

Mr. R. P. HARDY—Mr. Sprague, I believe, after the lengthened study he has given to this question, has been unable to suggest a method more generally applicable than that which is ordinarily adopted. In some cases a low rate of premium during the joint lives of the successive tenants can be charged; and in others, no doubt, an office would be disposed to grant an increasing assurance for a limited term. The particular case which Mr. Sprague instances must be the one case in a hundred. Borrowers always object to the advance being made by way of reversionary charge, and solicitors are very loth to take upon themselves the responsibility of advising a client to enter on a transaction which may ultimately prove detrimental to his interest. They all, therefore, naturally endeavour to make you entertain the transaction by way of accumulation. I have seen some such cases so carried out, and I think they will be found to be safely based, but they are very difficult indeed to manage, and the personal covenant of the borrower must be of some value. [Mr. BAILEY, "That is very rare."] Mr. Bailey thinks there is a difficulty in representing these transactions in the books of account, so as not to show a falling off in the interest revenue. I do not see that that ought to be the case. If you have a fair number of transactions of about equal amount, it is probable that a sufficient number will fall in in every quinquennium to justify the office allowing, say, 5 per-cent every year on the account. That is the case with offices dealing in moderate-sized reversions, and they will be found to fall in with some regularity; and if you look at the profit made, you will find it comes as nearly as possible to the rate of interest at which you have valued. Mr. Bailey says further, that most of these borrowers are tenants in tail, but I should say that eight out of every ten borrowers are tenants for life. The conveyancer, in framing a settlement, takes good care to give a tenant in tail no allowance out of the estate, so that when

he comes of age he has nothing to live upon, and his only means of obtaining a subsistence is by consenting to bar the entail, and to convert his estate into one for life merely. In the less frequent case of a nephew, who is in succession to an uncle, he is no doubt a tenant in tail; or at the best has only a base fee; and therefore it is that in most of these cases you have to provide for a whole-term assurance; but where the income is sufficiently large, you may rely upon the formula Mr. Sprague has given, taking receivership of the rents, and recouping yourselves out of income. I have drawn up a formula which may be conveniently used in those cases where you desire to make a loan by way of accumulation. It is, of course, more expensive in the end, but I believe it satisfies the prejudices of the borrower, who always imagines you are going to take advantage of him. The formula is—

$$1-(\mathrm{P}+d)(1+a_{\overline{n-1}|}+{}_{n-1|}a_{xy})$$

It represents the amount that can be advanced for every £1 of assurance. You retain in hand the value of an annuity certain of $\mathrm{P}+d$ for the term agreed on, and you further reserve an annuity-due of $\mathrm{P}+d$ upon the joint lives of the successive tenants deferred n years. By the end of n years the debt has grown exactly to the amount of the assurance, and there then becomes constituted an annuity-due of $\mathrm{P}+d$ for the remainder of the joint lives, which will prevent the debt growing to any further extent. The effect of this method may be illustrated by taking the case mentioned in the paper, where the ages of the lives are 60 and 30. For every £1000 advanced by way of surcharge, the fixed sum to be paid would be, say, £3480. By the method of account for, say, twelve years, the total advance grows by stages to £5060 as its ultimate amount, the less amount that would be paid in the event of the early falling in of the reversion compensating for the excess over the fixed charge which would become due in the event of the transaction outlasting the average term.

Mr. A. Baden—Mr. Sprague has said that these transactions form very desirable investments for insurance companies, and so no doubt they do in the majority of cases. But there is one caution which should be attached to them: that we should not lose sight of the consideration of the character of the reversionary life tenant. In the kind of transactions that assurance companies have to carry out, this consideration is generally reduced to a minimum, the families in most cases being honourable and of good repute, and may be considered in a great measure answerable for every one of their members. But occasionally cases do arise, of which we have two remarkable instances before us in the case of the claimant to the Tichborne estates and the heir to the Aberdeen peerage, where, either from vice on the part of the expectant, or from eccentricity, or both causes combined, there is great danger of the reversionary life tenant getting out of the way and concealing his whereabouts, and in consequence creating great difficulty on the part of the office in proving his death, or, in fact, in knowing what to do at all.

The President—In the way which I think Mr. Jellicoe first suggested of dealing with these questions, it was supposed that the lender purchased an annuity during the joint lives sufficient to pay premiums and interest. I never heard that that principle was ever carried out

in practice; but if it were carried out it would meet Mr. Bailey's difficulty as regards book-keeping. There is a difficulty no doubt in dealing with these questions in the ledger. You must credit a certain amount of interest. Whether it is prudent to take credit for the whole interest you are expected to receive, is doubtful. For myself, in a case of this sort where money has been advanced at 5 per-cent, I have never taken credit for more than 4 per-cent. If it should so happen that the reversion falls in early, there is something to the credit of profit and loss. The method suggested at the end of the paper is, I suppose, with the view of making the transaction less costly than by the ordinary way, and Mr. Hardy seems to have hit upon a plan of further reducing the cost. There is no doubt that a borrower is always staggered—or, at all events, his solicitor is—at the amount of the reversionary charge. To meet that difficulty, I have in two or three cases offered to do it simply as a mortgage by way of accumulation; but, in that case, you must take care to have a larger assurance, I think, than what you would otherwise provide for, because, as you can gain nothing by that means but your rate of interest, you cannot afford to run any risk. In all these transactions it is very important that we should look to the character of the borrower, because if he goes beyond the limits allowed by the policy, it becomes a serious matter, unless it is provided for in the first instance; and if it is provided for, it helps to make the transaction more costly.

Mr. SPRAGUE—Mr. Bailey has referred to the advantage we should derive if we had here what he calls the "counsel for the borrower," to examine the question from a borrower's point of view; but I think that, to a considerable extent, the line I have taken obviates the necessity for that. I have pointed out that we are to consider first of all, as a mathematical problem, what are the very lowest terms on which we can undertake these transactions. It is a similar case to that of a man engaged in large contracts—ship-building, for instance,—he first of all endeavours to ascertain what will be the actual cost of building the ship, and then he considers what he shall add for his profit on the contract; just in the same way, I consider what will be the actual cost to a company of going into one of these reversionary transactions, and then I charge something more for profit. The question as to the way of treating these advances, as regards the book-keeping, is one which I did not think it necessary to touch upon. My own practice has been to charge no interest at all on the accounts of reversions purchased. I find, as a matter of fact, that when a valuation comes round, if I have debited a reversionary annuity simply with the premiums paid, the cost in the ledger exceeds the value of the reversionary annuity, so that it has to be reduced, and if interest had been added, a still larger reduction would be necessary; and if it should happen in the short period of five years that out of ten or twelve large reversions purchased not one had fallen in, there might be a large amount of interest to be written off, which would look a very awkward item. Of course if an office buys absolute reversions, they always increase in value, and that increase in value may be set against the reduction in value of the reversionary annuities. But if all the reversions, both absolute and contingent, have been debited with interest every year, it must sometimes happen that there is no

source whatever out of which to provide for the reduction in value of reversionary annuities. In the office I am at the head of, I debit no interest at all; and then when a reversion does fall in, there is a considerable amount of profit which is available for any purpose required, or which may be simply carried into account as balance of profit and loss. With regard to the two classes of borrowers mentioned by Mr. Bailey, it will be noticed that my paper is confined to the consideration of a reversionary life interest, properly so called. A tenant in tail does not properly come under the same class as a borrower who has only a reversionary life interest, because we know that by legal arrangements lawyers can give us a charge upon the fee simple, provided the tenant in tail should survive the tenant for life.

The PRESIDENT—What you call a "base fee."

Mr. SPRAGUE—Yes, the tenant in tail can at once, by executing a disentailing deed, obtain a "base fee," or an interest in the estate, which lasts so long as he or any of his descendants (or male descendants if the estate is settled in tail male) shall be living—subject, of course, to the interest of the life tenant; and when the tenant for life dies, the tenant in tail, by executing another deed, can acquire the fee simple of the property. The practical means by which a borrower's "base fee" can be most effectually made a security for an advance is a matter of so much interest that I may not, perhaps, be occupying your time too long if I describe it. The borrower, having cut off the entail and acquired a base fee, executes a power of attorney, authorizing the lender and his solicitor, and perhaps half-a-dozen of his clerks, or any one of them, as soon as the tenant for life is dead, to execute the necessary further disentailing deed. Then that deed is prepared, and as soon as ever intelligence is received of the death of the life tenant, one of the persons named in the power of attorney executes the deed, and the thing is done without any further consent on the part of the borrower being necessary. Several names must be inserted in the power of attorney to provide against the risk of the attorney dying before the life tenant. The risk of the borrower dying before the life tenant is met by effecting insurances on his life against that of the life tenant; and the very remote risk of his dying after the life tenant, but before the disentailing deed is executed (as, for instance, both being drowned by the upsetting of a boat, and the younger man surviving the elder for a quarter of an hour), is met by making the risk under the contingent policies last for, say, three months after the death of the life tenant.

With reference to the formula—

$$1-(\mathrm{P}+d)(1+a_{xy})$$

Mr. Bailey remarks that it is Mr. Jellicoe's formula, and only applies when an annuity is actually bought; but if he looks at it a little closer he will find it is exactly the same formula as is really used in the process he describes, the only difference being in the rate of interest: In Mr. Jellicoe's method, d is taken at 5 per-cent, and the annuity on the joint lives is taken at $3\frac{1}{2}$ per-cent, say; but if you take d at 6 per-cent, and the annuity on the joint lives also at 6 per-cent,—that is exactly the same thing as the ordinary formula for the value of a reversion at 6 per-cent. For

$$\mathrm{A}_{xy}=1-d\,(1+a_{xy}),\text{ and }\mathrm{A}'^{1}_{xy}=\mathrm{P}'^{1}_{xy}\,(1+a_{xy}).$$

It is curious to note how the same identical formula will bring out different results, the difference being in the rate of interest. If, for instance, in the above formula P were taken as the net premium, the formula would be the absolute bare value of the reversion at 6 per-cent. The case which Mr. Bailey has mentioned, of the office undertaking to grant a whole-life assurance on the death of the tenant for life, whatever may then be the state of the borrower's health, is one which he will see I have fully considered in the paper. I now pass on to consider what Mr. Macfadyen has said. He does not quite see, I think, the use of the optional increasing assurance I have proposed. If you have an ordinary increasing assurance, no doubt if the borrower wants the insurance, the cost of it to him beyond the net premium is less than that of the optional increasing assurance; but the fact is, in many cases he does not want the increase of the insurance at all; and it is not, as Mr. Macfadyen supposes, like having a picture given him for part of his loan, for if the picture is a good one it will rather increase in value in time, but if a man has a large increasing assurance, he pays his premiums for the current risk, and has nothing to show for them afterwards. So that when a man is forced to effect such an assurance, which he does not want, it is no answer that it is an increasing assurance and he has had the benefit of it. He is exactly in the position of a man who is obliged by a money-lender to take a large quantity of wine that he does not want, which he cannot sell, which will turn sour if not drunk, and which he is therefore constrained to give away to his friends. By effecting this optional increasing assurance on the contrary, he knows the worst of it, and he has to pay a certain consideration, 5*s*. per-cent per annum. The company gets that consideration; the man pays it and does not grudge the money, because he gets the insurance he wants, and is saved the larger outlay for the increasing insurance which neither he nor the office wishes for, and which is of no value to anybody except his heirs if he should chance to die before the loan is paid off. So that the optional increasing insurance appears to me to give exactly what is wanted, and at the lowest possible price. The question mooted by Mr. Baden as to the character of the borrower is no doubt one of great importance, and has to do not only with reversionary life interests, but with life interests in possession. It is necessary in dealing with both of them to take care that you leave a sufficiently large margin to make it worth the while of the borrower to come forward from time to time and claim the surplus. If proper care is taken to have a margin, you may be sure, when the tenant for life dies, the reversioner will come forward and claim his interest. Mr. Tucker has told us that he never heard of the case of an annuity being actually bought. I do not know how it may be with regard to reversionary annuities, but I am informed there is one large old insurance company in the city that always buys an annuity when it buys a reversion, and thus entirely gets rid of the difficulty as to the interest account.

Mr. BAILEY—Buys it from itself?

Mr. SPRAGUE—No, it does not grant annuities; but actually purchases the annuity elsewhere.

On the Arithmometer of M. Thomas (de Colmar), and its application to the Construction of Life Contingency Tables. By PETER GRAY, F.R.A.S., F.R.M.S., *Honorary Member of the Institute of Actuaries.*

THE Arithmometer of M. Thomas (de Colmar) has been already brought under the notice of the readers of this *Journal* by General Hannyngton, in a remarkably lucid and suggestive paper, which will be found at p. 244, vol. xvi. General Hannyngton, in his paper, explains the manner of working the machine, and gives examples of some of its applications to the construction of actuarial tables, with hints as to others. These afford an idea of the very striking adaptation of the machine to the formation of such tables; and they cannot fail to have excited the interest of many of the readers.

The present paper is intended to be supplemental to that of General Hannyngton, and in it the adaptation in question will be further shown. An attempt will also be made to systematize the manner of its application, and detailed examples in illustration will be given. There will be no need to say anything here as to the actual working of the machine, since this has been so well explained by General Hannyngton. There are, nevertheless, certain points in the manipulation to which it may be well to advert in the outset.

It is usual to describe the Arithmometer as a machine which enables a person, however unskilled himself, to perform the operations of multiplication and division with facility, rapidity, and unfailing accuracy. This, as a description, is correct as far as it goes; but as an enumeration of the properties of the machine, it is inadequate and defective. It entirely omits that property which forms its special adaptation to our purpose, and in default of which its utility would be comparatively limited. Besides the facilitation of the operations named, the machine will also, in forming the product of two given numbers, either add that product to, or subtract it from, another given number, according to the pleasure of the operator. Abundant illustration of the application of this property* will be found in the present paper.

There are three forms, then, to the numerical evaluation of which the Arithmometer is directly applicable. These may be symbolized as follows:—

$$\text{QR}, \quad \frac{\text{Q}}{\text{R}}, \quad \text{and } \text{P} \pm \text{QR};$$

P, Q and R denoting any given numbers.

* An illustration of it is given by General Hannyngton, in the formation of $\text{D}_{x.y}$.

Familiar instances of these forms, in the present connexion, are:—

$$\text{QR} \quad . \quad . \quad . \quad . \quad . \quad D_x = l_x v^x;$$

$$\frac{\text{Q}}{\text{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad a_x = \frac{N_x}{D_x};$$

$$\text{and} \quad \begin{cases} \text{P}+\text{QR} \quad . \quad . \quad . \quad . \quad N_x = N_{x+1} + l_{x+1} v^{x+1}; \\ \text{P}-\text{QR} \quad . \quad . \quad . \quad . \quad A_x = 1-(1-v)(1+a_x). \end{cases}$$

The manner of applying the machine to the evaluation of expressions of the first and second forms; in other words, the manner of performing the operations of multiplication and division, has been sufficiently explained in General Hannyngton's paper. Of its application to the third form some further elucidation is necessary.

The special adaptation of this form to the construction of tables may be shown as follows:—

If u_x be any function of x, we always have,

$$u_{x+1} = u_x + \Delta u_x;$$

and this is of the form $\text{P} \pm \text{QR}$ if Δu_x, the difference of u_x, can be exhibited as the product of two given numbers. In such cases, then, (and it will soon appear that they are by no means rare), we have, in forming a series of successive values, the benefit of a continuous process, with its attendant advantages. The result of each operation becomes the P of the next, and we pass from a preceding to a succeeding value by adding to (or subtracting from) the former the product of two known numbers. And this, as we have seen, is a function to which the Arithmometer most readily lends itself.

In the applications of the machine there are thus, it appears, three numbers to be dealt with; and provision is accordingly made on it of three spaces for their separate exhibition. Two of these spaces are on the *slide*, and the third on the *face*. I propose to designate them by S_1, S_2, and F, respectively; and the numbers occupying the several spaces will be denoted by the same symbols, respectively distinguished, however, when so used, by being enclosed in parentheses, thus:—(S_1), (S_2), (F).

In employing the formula $\text{P} \pm \text{QR}$, the number P is placed on S_1, and Q (the *in* factor), upon F; multiplication being then made by R (the *out* factor), this number appears upon S_2, and the result, which is the required value of $\text{P} \pm \text{QR}$, replaces P upon S_1.

A point in regard to which difficulty is felt, in commencing the use of the Arithmometer, is the setting of the numbers P and Q upon the machine, so that the product QR shall, when formed, fall

in its proper place with respect to P. The rule for this purpose (which has not hitherto been given) is very simple. It is:—

Draw out the slide as many holes as there are decimal places in R, the *out* factor; and place P and Q upon S_1 and F respectively, in such a manner as that like denominations shall stand under (and over) like. The slide is then to be pushed home; and, the multiplication being performed, the correct result will appear on S_1.

If $P=0$, that is, if it is only the product QR that is required, no preliminary drawing out of the slide is necessary.

So far as the results of the formulæ QR, $P \pm QR$ are concerned, it is obviously indifferent which of the two, Q, R, we employ as the *in* factor; in each case of practical application, however, there are usually circumstances sufficient to determine our choice in favour of one or the other. I shall in general, when necessary for distinction, let Q denote the *in* factor—the factor to be set upon F; and R, of course, to denote the *out* factor—the factor to be employed as a multiplier.

The points that have been now more specially adverted to, and others that may arise, will find ample illustration in the examples appended to the problems to which I now proceed.

The examples will be taken from *The Institute of Actuaries' Life Tables*, H^M mortality, and three per-cent interest.

PROBLEM I.—Given a table of annuities; to form the corresponding table of assurances.

We have, $$A_x=1-(1-v)(1+a_x) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \quad \Delta A_x=-(1-v)\Delta a_x;$$

whence, $$A_{x+1}=A_x+\Delta A_x=A_x-(1-v)\Delta a_x \quad . \quad . \quad . \quad . \quad (2)$$

By these formulæ the required table will be constructed, (1) coming into use for the formation of the initial term, and (2) for the continuous formation of the subsequent terms.

Example 1.—The given table of annuities is that on single lives, p. 14 (*Institute Tables*); and it is required to reproduce the table of assurances in the adjoining column.

Here we have, by (1),

$$\begin{aligned} P &= \quad 1{\cdot}0000000, \\ Q &= 1-v = {\cdot}0291262, \\ R &= 1+a_{10} = 25{\cdot}1484. \end{aligned}$$

We here take $1-v$ for Q, because it remains constant during the process; and we use six significant figures because, there being

also six in R, we desire to have our results true to the same extent.

P and Q are placed on the machine as follows:—

$$\begin{array}{ll} S_1 & 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \,|\, 0\,0\,0\,0 \\ F & \qquad\quad 0\,0\,2\,9\,1\,2\,6\,2 \,| \end{array}$$

There are *four* decimal places in R; *therefore*, by the rule, the denominations of the figures in P and Q are made to correspond when the slide is drawn out *four* holes.

One of the ivory pegs which accompany the machine is now inserted in the place of the decimal point, and another between the sixth and seventh decimal places; the last-named place also (the first beyond the peg) is increased by 5. The pegs serve to direct the eye to the part of the slide where the figures to be taken out for record are to be found; and the effect of the addition of 5 in the *next* place being to impart to the adjoining figure the usual correction, when necessary,* the eye, in taking out the result, has never to travel beyond the space embraced by the two pegs.

I repeat the representation of the setting of the numbers on the machine for the present example, adding indications of the positions of the pegs and of the increased figure:—

$$\begin{array}{r} 0\,0\,0\,0\,1,0\,0\,0\,0\,0\,0,\overset{*}{5} \,|\, 0\,0\,0\,0 \\ 0\,0\,2\,9\,1\,2\,6\,2 \,|\, \end{array}$$

The same indications will be used in subsequent representations of the setting of the machine.

When the slide is pushed in as directed, the above setting will assume the following appearance:—

$$\begin{array}{r} 0\,0\,0\,0\,1,0\,0\,0\,0\,0\,0,\overset{*}{5}\,0\,0\,0\,0 \,| \\ 0\,0\,2\,9\,1\,2\,6\,2 \,| \end{array}$$

The form here being P−QR, and both the factors Q, R, being positive, the regulator is set for subtraction; multiplication is then made by R=251484, and the result is,

$$0\,0\,0\,0\,0,2\,6\,7\,5\,2\,3,\overset{*}{1}\,7\,1\,9\,2.$$

* The method here made use of for the correction of the last figure in the terms of series formed by addition (or subtraction) was, I believe, first suggested and exemplified by me, in a work published in 1849. Among the instances of its advantageous employment that have come to my knowledge, the most striking is one that occurred to Dr. Farr, in the construction of his tables of $\log v^x$ (*Tables of Life-Times*, pp. 6–11). These tables were formed by the aid of Scheutz's machine; a specialty of which is, that it records its results in the form of a mould to be employed as the matrix for a stereotype cast. It is consequently necessary, in the use of this machine, that each result, *as it arises*, should be correct in the last figure to be recorded. Dr. Farr informs us, in his Introduction (p. cxliii), that the method he employed for this purpose was that now under consideration. The required correction could probably, in this case, have been given in no other way.

Had the increase of 5 in the fifth place from the last not been made, the figures which now read 3,1, would have read 2,6; and in transcribing the result we should have had to correct it in the last place. The correction made in this manner once for all, suffices for the whole of the succeeding values; and, as has been already said, we shall not have occasion to look at any figures except those in the space between the pegs.

The expression (2) now comes into use for the completion of the series. It is,

$$A_{x+1}=A_x-(1-v)\Delta a_x;$$

which when $x=10$ becomes,

$$A_{11}=A_{10}-(1-v)\Delta a_{10}.$$

This also is of the form $P-QR$; but here R, that is Δa_{10}, being negative, the regulator must be set for addition. In other respects the setting of the machine remains; and after effacing (S_2), the series is completed by employing as multipliers the successive terms of Δa_x, effacing each, of course, after it has been used.

The series Δa_x should be formed on a separate slip of paper or cardboard (and proved by addition), as it comes into use in other formations.

The commencing portion of the results, as they come out, is here given; and it may be compared with the corresponding portion of the series* on p. 14 of the Institute volume.

x	Δa_x	A_x
10		·267523
11	1531	·271982
12	1811	·277257
13	2039	·283195
14	2206	·289621
15	2316	·296366
16	2366	·303258
17	2354	·310114
18	2284	·316766
19	2148	·323023
20	2004	·328860
	1941	
*	*	*

* The comparison here suggested will reveal the existence of a few slight discrepancies in the last place between the two sets of values compared. Both nevertheless are correct deductions from the data employed in their formation. The discrepancies, such as they are, originate in the relation which subsists between the values of an annuity and an assurance on the same status. It can easily be shown that, *at three per-cent*, four-decimal annuities are barely sufficient for the accurate determination of six-decimal assurances.

Although the machine (when in order) does not commit mistakes, the operator enjoys no such immunity from error. He may, for instance, have used a wrong figure in one of the multipliers;* and the error thus committed will, unless discovered and corrected, vitiate the succeeding values. It is, therefore, necessary to be provided with means for its speedy detection. These will be three or four terms of the series, at equal intervals, which may be readily formed in the same manner as the initial term. Each of these, as it is reached, will form a check on all the preceding work.

The series A_x may be formed with the same facility if we commence the formation with the oldest instead of the youngest tabular age. The chief difference in the working will be, that as in this order Δa_x is positive, the regulator will remain at subtraction throughout. The initial term will be

$$\begin{aligned} A_{97} &= 1-(1-v)(1+a_{97}) \\ &= 1-(1-v)\times 1{\cdot}0000, \end{aligned}$$

since $A_{97}=v$; and it will be formed, after setting on P and Q as for A_{10}, by a single turn, *without pushing in the slide.*

Example 2.—Let the given annuity series be that in column 64, p. 150; to form the corresponding assurance series.

The process here is in such entire analogy to that exemplified in Example 1, that it will be sufficient, after two remarks, to give a specimen.

The first remark is, that while, as a rule, the differences of the annuities are negative, there are exceptions; as here $\Delta a_{64\cdot 10}$ is positive. Differences thus abnormally affected should be written in red ink; and this will serve as a warning, when using them, to alter the regulator from its normal position.

The second remark is, that, whereas the differences of the single-life annuities consist generally of four significant figures, those of the joint-life annuities, as here arranged, have never more than three. The assurances corresponding to these will therefore admit of construction at the cost of a proportionally less amount of labour. I find, in fact, that a column (45 values), after the differences and verifications have been formed, need not occupy in construction (including the recording), more than twenty minutes. It would hence be quite a practicable task to form the assurances

* The facility with which errors of this kind, *when noticed at the time,* admit of correction, forms one of the great merits of the Arithmometer. The erroneous figure being brought into the working position, it is set right by the requisite number of turns; remembering that, preparatory thereto, if it is diminution that is required, the regulator must be reversed.

corresponding to the joint-life and last-survivor annuities in the Institute volume.

The following is the specimen above referred to :—

x	$\Delta a_{64.x}$	
10		·723150
11	32	·723056
12	58	·723225
13	136	·723621
14	196	·724192
15	239	·724888
16	263	·725654
17	264	·726423
18	238	·727117
19	186	·727658
20	121	·728011
	87	
*	*	*

PROBLEM II.—A table of annuities being given, it is required to construct a table of the Values of Policies at all ages, and for all durations.

The technical "Value of a Policy" is the difference between the value, on an anniversary of its inception (and consequently when a premium is just due), of the sum assured (supposed a unit) and the premium payable in respect of it, on the supposition that this is the *net premium* according to the table used in the valuation.

Denoting, for the present purpose, the value of a policy on a life now aged x, which was effected at age w, by $V_{w]x}$*, it is known that,

$$V_{w]x}=1-\frac{1+a_x}{1+a_w}$$

This may be written,

$$V_{w]x}=1-(1+a_w)^{-1}.(1+a_x);$$

so that by using the reciprocal of $1+a_w$ the indicated division is changed into a multiplication. The expression is now of the form P$-$QR; and it is consequently fitted for the application of the Arithmometer.

The given annuity table being, as before, that on p. 14, the following scheme will facilitate the comprehension of the order of construction :—

* I find that the symbol for the value of a policy, ${}_nV_x$, given in the recognized notation, is unsuited when, as in the present investigation, it is necessary to treat the value as a function of the ages at entry and at valuation. I have therefore been obliged to devise a new symbol; and the one in the text, while answering my present purpose sufficiently well, is so distinctive that there is no risk of its being confused with any other symbol.

x	10	11	12	13	14	15	16
10	10·10						
11	11·10	11·11					
12	12·10	12·11	12·12				
13	13·10	13·11	13·12	13·13			
14	14·10	14·11	14·12	14·13	14·14		
15	15·10	15·11	15·12	15·13	15·14	15·15	
16	16·10	16·11	16·12	16·13	16·14	16·15	16·16

Here the values of x, the present age, are at the side, and those of w, the age at entry, at the top; so that w is constant in the columns, and x in the rows.

We have, $\quad V_{w]x}=1-(1+a_w)^{-1}.(1+a_x)$ . . . . (3)

It is most convenient to effect the construction in columns. Hence, making x the variable,

$$\Delta_x V_{w]x}=-(1+a_w)^{-1}.\Delta a_x;$$

and, $\quad V_{w]x+1}=V_{w]x}-(1+a_w)^{-1}.\Delta a_x$ . . . (4)

The similarity of these expressions to those that arose in Problem I. is apparent; the only difference being that in these $(1+a_w)^{-1}$ takes the place of $1-v$ in the others. The process of construction, accordingly, in each column, after the formation of the initial term, is absolutely identical with that of Problem I.

The initial term for the first column is

$$V_{10]10}=1-(1+a_{10})^{-1}.(1+a_{10});$$

so that we have,

$$\begin{aligned} P &= \quad 1{\cdot}0000000, \\ Q &= (1+a_{10})^{-1} = {\cdot}0397640, \\ R &= 1+a_{10} = 25{\cdot}1484; \end{aligned}$$

and the setting of the machine will be as follows :—

$$\begin{array}{l|l} 0\,0\,0\,0\,1{,}0\,0\,0\,0\,0{,}\overset{*}{5}\,0 & 0\,0\,0\,0 \\ \qquad\qquad 0\,0\,3\,9\,7\,6\,4\,0 & \end{array}$$

From (3) it appears that when $x=w$, $V_{w]x}=0$. That is, the initial term in all the columns is 0; and we might therefore assume this value, and commence the construction in each column with the formula (4). It is better, however, to proceed regularly, as we should not otherwise have that perfect identity which ought to subsist, between the values successively formed and those formed for verification in the manner to be presently shown. The reciprocals are only approximately, not absolutely true; and $V_{w]x}$ when $x=w$

never comes out exactly equal to 0, although, using the reciprocals to six significant figures, the deviation will in no case affect the *sixth* place of our results by more than a unit.

The numbers being placed on the machine as above, pushing in the slide, setting the regulator for subtraction, and multiplying by 251484, we get for $V_{10]10}$,

$$00000,00000,\overset{*}{4}02240.$$

The regulator is now changed to addition, Δa_x being negative throughout, and multiplication being made by the terms of this series in succession, commencing with $\Delta a_{10}=1531$, column 10 is completed.

The other columns are formed in the same way, the initial values of w and x in each succeeding column being respectively greater by unity than in the column preceding.

The following is a specimen of the formation, embracing the commencement of the first six columns.

		397640	400075	402995	406334	410009	413940
w	Δa_x	10	11	12	13	14	15
10		·00000					
11	1531	·00609	·00000				
12	1811	·01329	·00725	·00000			
13	2039	·02140	·01540	·00822	·00000		
14	2206	·03017	·02423	·01711	·00896	·00000	
15	2316	·03938	·03349	·02644	·01837	·00950	·00000
16	2366	·04879	·04296	·03598	·02799	·01920	·00979
17	2354	·05815	·05238	·04546	·03755	·02885	·01954
18	2284	·06723	·06152	·05467	·04683	·03821	·02899
19	2148	·07577	·07011	·06332	·05556	·04702	·03788
20	2004	·08374	·07813	·07140	·06370	·05524	·04618
	1941						
*	*	*	*	*	*	*	*

The reciprocals* that come into use in the formation of the several columns, being those of $1+a_{10}$, $1+a_{11}$, &c., are here, for illustration, written at the top of the respective columns; and the slip containing the series Δa_x is represented by the side of column 10. It may, as each column is completed, be moved forward to the next.

And now, as to the formation of verifications. Resuming the expression (3)

$$V_{w]x}=1-(1+a_w)^{-1}.(1+a_x)$$

if in this we make w the variable, the values indicated will be those

* The reciprocals for the entire annuity column had better be formed at the outset, either by the machine or by Oakes's Table.

occupying the row opposite x. We may therefore, by choosing three or four values of x at suitable intervals, and forming the corresponding horizontal series, thus obtain the requisite values for verification of the work in the columns. Taking the difference with respect to w, therefore, we have,

$$\Delta_w V_{w]x} = -(1+a_x).\Delta(1+a_w)^{-1};$$

whence, $$V_{w+1]x} = V_{w]x} - (1+a_x).\Delta(1+a_w)^{-1}. \quad . \quad . \quad (5)$$

Making $x=20$ and $w=10$, we have for the initial term of the row opposite 20,

$$V_{10]20} = 1-(1+a_{20})(1+a_{10})^{-1},$$

and for the next term

$$V_{11]20} = V_{10]20} - (1+a_{20}).\Delta(1+a_{10})^{-1};$$

the series being continued by using as multipliers the successive terms of the series $\Delta(1+a_w)^{-1}$, which is deduced from the values placed at the tops of the several columns in the last formation.

For the initial term:—

$$\begin{aligned} P &= \quad 1{\cdot}0000, \\ Q &= 1+a_{20} = 23{\cdot}0425, \\ R &= (1+a_{10})^{-1} = {\cdot}0397640; \end{aligned}$$

and P and Q are placed on the machine as follows:—

$$\begin{array}{r|l} 0\,0\,0\,0\,1{,}0\,0\,0\,0 & 0{,}\overset{*}{5}\,0\,0\,0\,0\,0 \\ 0\,0\,2\,3\,0\,4\,2\,5 & \end{array}$$

There being *seven* decimal places in R, the denominations in P and Q are made to correspond when the slide is drawn out *seven* holes.

Pushing in the slide, setting the regulator for subtraction, and multiplying successively by $(1+a_{10})^{-1}=397640$, and the differences of this series, the terms in line with $x=20$ come out as follows:—

w	$\Delta(1+a_w)^{-1}$	$V_{w]20}$
10		·08374
11	2435	·07813
12	2920	·07140
13	3339	·06370
14	3675	·05524
15	3931	·04618
16	4094	·03675
17	4156	·02717
18	4109	·01770
19	3940	·00862
20	3742	·00000
	3686	
*	*	*

The whole of the work in Columns 10 to 20, down to $x=20$, is thus verified. And in the same way verification may be obtained at as many points as we please, by forming the requisite horizontal series. These ought to be formed first, and inserted in their places, in order that, if error be committed in any of the columns, it may be arrested, and not suffered to proceed beyond the next point of verification.

It may be pointed out that the last horizontal series,—that corresponding to $x=97$,—which will serve as a final verification of all the columns, may be formed most conveniently without the aid of the machine, as follows:—When $x=97$, the expression for the value of the policy becomes,

$$V_{w}]_{97}=1-(1+a_{w})^{-1};$$

and the series of final terms will be formed in order, by subtracting from unity, continuously, $(1+a_{10})^{-1}$ and the differences of the series of which this is the first term. Thus:—

	$V_{w}]_{97}$
	*
	1·0000050
	0397640
10	·9602410
	2435
11	·9599975
	2920
12	·9597055
	3339
13	·9593716
	3675
14	·9590041
	3931
15	·9586110
	4094
16	·9582016
	4156
17	·9577860
*	*

The last term of this series ($x=97$, $w=97$), like those of all the other horizontal series, will come out equal to 0.

There is another arrangement of the table of the Values of Policies which is usually preferred. In it the values of w are

still at the top; but the argument at the side, instead of x, is $x-w$, the duration of the policy. The following is the commencement of the table according to this arrangement.

$x-w$	10	11	12	13	14	15	$x-w$
0	·00000	·00000	·00000	·00000	·00000	·00000	0
1	·00609	·00725	·00822	·00896	·00950	·00979	1
2	·01329	·01540	·01711	·01837	·01920	·01954	2
3	·02140	·02423	·02644	.02799	·02885	·02899	3
4	·03017	·03349	·03598	·03755	·03821	·03788	4
5	·03938	·04296	·04546	·04683	·04702	·04618	5
6	·04879	·05238	·05467	·05556	·05524	·05421	6
7	·05815	·06152	·06332	·06370	·06320	·06217	7
8	·06723	·07011	·07140	·07159	·07108	·07031	8
9	·07577	·07813	·07922	·07940	·07914	·07882	9
10	·08374	·08589	·08697	·08739	·08757	·08776	10
*	*	*	*	*	*	*	*

The computation can be conducted in this form just as easily as in the other. The only changes in the process will be, that the slip containing the differences, while being carried forward from each column to the next, will have to be raised one line; and that the series formed for verification will take their places, not in horizontal lines, but in ascending diagonal lines.

PROBLEM III.—To construct Columns N_x and D_x, l_x and v^x being given.

It is advisedly that I here place N_x before D_x, the construction in this order being the easier of the two.

We have, $\quad N_x=D_{x+1}+D_{x+2}+\ldots.;$

whence, $\quad \Delta N_x=-D_{x+1}=-v^{x+1}l_{x+1},$

and, $\quad N_{x+1}=N_x+\Delta N_x=N_x-v^{x+1}l_{x+1}.$

By aid of this expression, which is of the form P−QR, the column might be constructed, if we had the means of determining independently N_x for the youngest tabular age, namely 10 years. But this we have not. We therefore commence with the oldest age, and writing the above

$$N_x=N_{x+1}+v^{x+1}l_{x+1},$$

we have for an initial term, N_{97} being$=0$,

$$N_{96}=v^{97}l_{97}.$$

The method of forming this is obvious.

For the remainder of the column, the expression being now of the form, P+QR, we have $P=N_{x+1}$, $Q=v^{x+1}$, and $R=l_{x+1}$; and the manner of proceeding here, too, is obvious.

It will be observed, however, that the operation here differs in an important respect from those of the preceding problems. In them Q (the *in* factor) being constant, the setting on F remains unchanged till the completion of the column; here, however, $Q(=v^{x+1})$ varies with x, and the setting has to be altered for each term. The operation is nevertheless still continuous, inasmuch as each result enters into, and forms part of, that which follows.

The following is a specimen of the process. It includes also the formation of D_x.

x	v^x.	l_x	N_x	D_x
	520 3284			
	535 9383			
97	552 0164		·00000000	·51171921
6	568 5769	9	·51171921	2·86960758
5	585 6342	49	3·38132679	8·14324320
4	603 2032	135	11·52456999	17·02360082
3	621 2993	274	28·54817081	30·01310627
2	639 9383	469	58·56127708	47·65556172
1	659 1364	723	106·21683880	71·42138460
0	678 9105	1052	177·63822340	102·09457340
*	* *	*	* * *	* * *
59	1697 3309	58866	112263·88510	10582·68657
8	1748 2508	60533	122846·57167	11186·83881
7	1800 6984	62125	134033·41048	11805·65929
6	1854 7193	63652	145839·06977	12439·12396
5	1910 3609	65114	158278·19373	13087·57478
4	1967 6717	66513	171365·76851	13751·57773
3	2026 7019	67852	185117·34624	14432·57755
2	2087 5029	69138	199549·92379	15131·09578
1	2150 1280	70373	214681·01957	15849·23394
0	2214 6318	71566	230530·25351	16589·31550
*	* *	*	* * *	* * *
19	5536 7575	96223	1227620·99017	55191·71170
8	5702 8603	96779	1282812·70187	57121·18885
7	5873 9461	97245	1339933·89072	59064·12591
6	6050 1645	97624	1398998·01663	61034·21644
5	6231 6694	97942	1460032·23307	63046·24818
4	6418 6195	98224	1523078·48125	65117·45981
3	6611 1781	98496	1588195·94106	67267·09717
2	6809 5134	98784	1655463·03823	69515·86405
1	7013 7988	99113	1724978·90228	71888·13958
0	7224 2126	99510	1796867·04186	74409·39100
	7440 9391	100000	1871276·43286	

The two columns following that containing the ages, represent two cardboard slips, on which are written, in reverse order, the terms of v^x and l_x, respectively. Their position with respect to each other and to the ages in the margin, is regulated in accordance with the formula of construction. Here we have, for example, opposite $x=96$, v^{97} and l_{97}; opposite $x=95$, v^{96} and l_{96}, and so on. The process now consists in the multiplying together of the corresponding numbers on the cards, in succession and the setting down of the results, as they arise, on the same line. These, as stated, are the terms of N_x corresponding to the values of x in the margin. It will be understood that in this process, in accordance with the formula of construction, the results are not removed from the machine, (S_2) alone requiring to be effaced after each multiplication.

I say nothing here as to the method of procuring verification of the foregoing process. I defer this, with further remarks on the construction, till the next following problem shall have been discussed.

Next, to form D_x. We have, as above,

$$D_x=-\Delta N_{x-1}=N_{x-1}-N_x.$$

Hence the Column D_x will be formed by differencing, in the usual way, the Column N_x, when written as above in reverse order. And the differences, when written as in the specimen, each opposite the subtrahend by the employment of which it is deduced, will be in their proper relation to the ages in the margin. And we have now, consequently, in line with x, v^{x+1}, l_{x+1}, N_x, and D_x.

To form D_x independently of N_x, the same process as that for the formation of the latter column would have to be gone through, with the addition that besides (S_2), (S_1) also would have to be effaced after each multiplication. The process also would be discontinuous; and in consequence, unless the work were performed in duplicate, no verification could be procured till both N_x and M_x should be in course of formation. If to this it be added that the deduction of D_x from N_x is at least as easy as that of N_x from D_x, sufficient will have been said to make it manifest that, in the construction of a Commutation Table by the aid of the Arithmometer, the proper course, in regard to the annuity columns, is to commence the formation with N_x. In the case of the assurance columns also, corresponding advantages attend the commencement of the formation with M_x.

Problem IV.—Given v^x and d_x; to construct Columns M_x and C_x.

Since, $$M_x = C_x + C_{x+1} + \ldots .$$

$$\therefore \qquad \Delta M_x = -C_x = -v^{x+1} d_x .$$

Hence, $$M_{x+1} = M_x + \Delta M_x = M_x - v^{x+1} d_x .$$

Column M_x, like Column N_x, and for a like reason, must be constructed in reverse order. Accordingly, transposing, we have,

$$M_x = M_{x+1} + v^{x+1} d_x ,$$

a formula for the construction in such entire accordance with that of the last problem, that little more is necessary than to direct attention to the specimen here given.

x	v^x	d_x	M_x	C_x
	520 3284			
	535 9383		·00000000	
97	552 0164	9	·49681476	·49681476
6	568 5769	40	2·77112236	2·27430760
5	585 6342	86	7·80757648	5·03645412
4	603 2032	139	16·19210096	8·38452448
3	621 2993	195	28·30743731	12·11533635
2	639 9383	254	44·56187013	16·25443282
1	659 1364	329	66·24745769	21·68558756
0	678 9105	408	93·94700609	27·69954840
*	* *	*	* * *	* * *
59	1697 3309	1667	7004·63125	282·94506
8	1748 2508	1592	7282·95278	278·32153
7	1800 6984	1527	7557·91943	274·96665
6	1854 7193	1462	7829·07939	271·15996
5	1910 3609	1399	8096·33888	267·25949
4	1967 6717	1339	8359·81012	263·47124
3	2026 7019	1286	8620·44398	260·63386
2	2087 5029	1235	8878·25059	257·80661
1	2150 1280	1193	9134·76086	256·51027
0	2214 6318	1160	9391·65815	256·89729
*	* *	*	* * *	* * *
19	5536 7575	556	17828·23462	307·84372
8	5702 8603	466	18093·98791	265·75329
7	5873 9461	379	18316·61047	222·62256
6	6050 1645	318	18509·00570	192·39523
5	6231 6694	282	18684·73878	175·73308
4	6418 6195	272	18859·32523	174·58645
3	6611 1781	288	19049·72716	190·40193
2	6809 5134	329	19273·76015	224·03299
1	7013 7988	397	19552·20796	278·44781
0	7224 2126	490	19906·19438	353·98642

The cards here are v^x and d_x, and their disposition, to accord with the working formula, is somewhat different from that of the cards in the last problem. Thus, in line with age x we have here v^{x+1} and d_x. When so arranged, the process for the formation of M_x is identical with that of last problem for the formation of N_x.

To form C_x,

$$C_x = -\Delta M_x = M_x - M_{x+1}.$$

Hence the column will be formed by differencing, as they stand, the terms of M_x; observing that here the differences as they arise are to be written, not as in the last problem, opposite the subtrahends, but opposite the minuends. When so written we have, in line with x, v^{x+1}, d_x, M_x, and C_x.

It has not been usual hitherto to tabulate C_x. But this column can be turned to such good account, for the formation of various subsidiary tables by the aid of the Arithmometer, that it will probably come to be tabulated, in a special category, along with others similarly adapted, such as Δa_x, D_x^{-1}, N_x^{-1}, &c.

When we multiply together two factors, of which one is a non-terminating number, a portion of the result has to be rejected, as not properly belonging to the product; since it would be altered if the non-terminating number were further extended. And therefore it is that in the process of contracted multiplication labour is saved by so arranging the work that only the *correct* portion of the product is formed. The Arithmometer attains the same end in another way: in the use of it we form the *entire* product of the numbers submitted to it, and we neglect the useless figures in recording the results. It is therefore desirable to be furnished with a guide as to the extent to which this process of curtailment ought to be carried.

In the case supposed—the multiplication of a terminating and a non-terminating factor—we know that we can depend upon about as many figures in the product as there are of significant figures in the non-terminating factor; and hence in the several terms of D_x and C_x, we shall have as many places true as there are in the terms of v^x which enter them, respectively. Also, in the several terms of N_x and M_x, which are respectively summations of series of terms in D_x and C_x, we should expect to find one or two more figures correct than in the corresponding terms of D_x and C_x.

We may say, then, that in the formation of N_x and M_x it is unnecessary to record any term to more than two places beyond the number of significant figures in the power of v in immediate use.

We shall find these conclusions verified in the case of the columns before us. I have used in the formation the column v^x (3 per-cent) as contained in Jones, vol. i, pp. 79 and 82, having first verified it by the aid of the Arithmometer. It contains seven significant figures at the outset, which further on are increased to eight; and I have used it to its full extent for experiment, although generally one or two places fewer will be considered sufficient.

Corresponding values in N_x and M_x are connected by the equation,

$$M_x = vN_{x-1} - N_x;$$

and applying this at a few points we find,

$$\begin{aligned} M_{91} &= \quad 66{\cdot}24745,576 \\ M_{59} &= \ 7004{\cdot}631,09 \\ M_{19} &= 17828{\cdot}234,95 \\ M_{10} &= 19906{\cdot}1939,3 \end{aligned}$$

The commas indicate the extent to which the values thus formed agree with those formed by the use of the machine. The agreement in the four examples extends to seven, eight, and nine places, respectively; but when we attend to the manner in which the formula lends itself to these deductions, we shall not fail to see that in each case the correctness of the Ns concerned is established generally to two places more. For example:—

$$\begin{array}{rl} 103)\overline{128281270187} & N_{18} \\ 124544922512 & vN_{18} \\ 122762099017 & N_{19} \\ \hline 17828234,95 & M_{19} \end{array}$$

Ten places in N_{18} and N_{19} we perceive here come into use for the determination of the eight correct places in M_{19}. We conclude, therefore, that at this part of the table N_x may be depended on to ten places of figures, and no more than this number need have been recorded.

Also, if the work has been correctly performed, there appears no reason why M_x should not be accepted as true to the same extent as N_x. As to the number of places to be finally tabulated

for use, the computer will, of course, follow the dictates of his own judgment.

By the use of the foregoing formula, N_x and M_x may be verified as the work proceeds. Both columns being brought down to the same point by the formation of, say, twenty terms in each, comparison can be made at this point as above shown. If found satisfactory, the formation may be continued till another point of comparison is reached; and so on till the columns are completed.

It is hardly necessary to point out that the contrivance formerly suggested for correcting the last figure retained in the several terms, cannot in these constructions, be advantageously applied; the reason being, that here the last figures vary in their distances from the decimal point.

The pegs also are employed here in a manner different from that in which they came into use in Problems I and II. They are now employed to facilitate the recording, by separating the results on S_1 into convenient periods.

(*To be continued.*)

Note on a Method of finding the Value of an Annuity on the Last Surviver of Three Lives.

MR. William Godward, of the Law Life Office, has communicated to us a method by which the values of annuities on the last surviver of two lives contained in the Institute Life Tables may be very conveniently applied to calculate the values of annuities on the last surviver of three lives.

Let x, y, z be the three lives, of which x is the youngest, and let w be the single age equivalent to the joint lives y and z (so that $a_w = a_{yz}$), then the value of the annuity on the last surviver of x, y, z may be found by the formula

$$a_{\overline{xyz}} = a_{\overline{xy}} + a_{\overline{xz}} - a_{\overline{xw}}.$$

The truth of this formula, which Mr. Godward informs us is given by Simpson in the Supplement to his Annuities, p. 58, is easily demonstrated.

We have

$$a_{\overline{xyz}} = a_x + a_y + a_z - a_{xy} - a_{xz} - a_{yz} + a_{xyz}$$
$$= (a_x + a_y - a_{xy}) + (a_x + a_z - a_{xz}) - (a_x + a_{yz} - a_{xyz}).$$

The two first of the three terms on the right hand side are the well known formulas for the values of annuities on two lives, $a_{\overline{xy}}$ and $a_{\overline{xz}}$. Also, since by supposition, $a_{yz} = a_w$, and therefore $a_{xyz} = a_{xw}$ approximately, the third of the above terms becomes $a_x + a_w - a_{xw}$ or $a_{\overline{xw}}$, whence the formula is proved.

The values of annuities on the last surviver of two lives according to the H[M] table being tabulated for all ages in the volume of tables published by the Institute of Actuaries, the above formula will give us the value of an annuity on the last surviver of three lives much more expeditiously than the ordinary formula, as is seen from the following example, in which the rate of interest is 3 per-cent:—

By Simpson's Formula.

$a_{40.50} = 11{\cdot}8177 = a_{56}$

$\therefore\ a_{\overline{30.40.50}} = a_{\overline{30.40}} + a_{\overline{30.50}} - a_{\overline{30.56}}$

$a_{\overline{30.40}}$	22·2274
$a_{\overline{30.50}}$	21·2947
	43·5221
$-a_{\overline{30.56}}$	−20·8569
	22·6652 $= a_{\overline{30.40.50}}$

By the Ordinary Formula.

a_{30}	19·8674	$a_{30.40}$	14·8162
a_{40}	17·1762	$a_{30.50}$	12·4690
a_{50}	13·8963	$a_{40.50}$	11·8177 $= a_{56}$
$a_{30.56}$	10·7347		39·1029
	61·6746		
	−39·1029		
	22·5717 $= a_{\overline{30.40.50}}$		

The two values, it will be seen, differ in the first decimal place; and this being the case, it is a matter of some importance to ascertain which of them is the more correct. In order to determine this, we have calculated the annuity by means of Mr. Woolhouse's formula, the process being exactly similar to that described and illustrated in vol. xvi p. 375, except that m has been in this instance taken equal to 11; and we thus got the value, 22·6174. As a further check, we have applied Mr. Woolhouse's formula to find the value of the annuity on the joint lives, $a_{30.40.50}$; and as the process is slightly different, we think some of our readers may be glad to see it explained at length.

We have

$$a_{30.40.50} = m\mathrm{S} + \frac{m-1}{2} - \frac{m^2-1}{12}(\mu_{30.40.50} + \delta)$$

where $$\mathrm{S} = {}_{m}p_{30.40.50}v^m + {}_{2m}p_{30.40.50}v^{2m} + \ldots.$$

Taking $m=11$, we get

$$a_{30.40.50}=11\mathrm{S}+5-10(\mu_{30.40.50}+\delta)$$

where $$\mathrm{S}={}_{11}p_{30.40.50}v^{11}+{}_{22}p_{30.40.50}v^{22}+\ldots.$$

Now ${}_{11}p_{30.40.50}=\dfrac{l_{41}.l_{51}.l_{61}}{l_{30}.l_{40}.l_{50}}$, and we have

$\log l_{41}$	4·910816	$\log l_{30}$	4·953591
„ l_{51}	4·854707	„ l_{40}	4·915315
„ l_{61}	4·756781	„ l_{50}	4·861690
	14·522304		14·730596
	−14·730596		

$\bar{1}$·791708 $=\log {}_{11}p_{30.40.50}$.

Proceeding in this way, the values of ${}_{11}p_{30.40.50}v^{11}$, ${}_{22}p_{30.40.50}v^{22}$, &c., were found, being those given in the last column of the following table:—

n	$\log {}_{n}p_{30.40.50}$	$\log v^n$	$\log {}_{n}p_{30.40.50}v^n$	${}_{n}p_{30.40.50}v^n$
11	$\bar{1}$·791708	$\bar{1}$·858791	$\bar{1}$·650499	·44720
22	$\bar{1}$·382154	$\bar{1}$·717581	$\bar{1}$·099735	·12582
33	$\bar{2}$·405367	$\bar{1}$·576372	$\bar{3}$·981739	·00959
44	$\bar{5}$·988925	$\bar{1}$·435162	$\bar{5}$·424087	·00003

·58264 = S
11

6·4090 = 11S
+5

11·4090

It remains to calculate $10(\mu_{30.40.50}+\delta)=10(\mu_{30}+\mu_{40}+\mu_{50}+\delta)$. The approximate formula for the value of μ,

$$\mu_x=\frac{d_{x-1}+d_x}{2l_x}$$

will not be sufficient for our present purpose, as we require to find the value of $\mu_{30.40.50}$ correct to five decimal places. We therefore use the more correct formula resulting from Mr. Woolhouse's formula (8) given in vol. xi p. 317:—

$$-\mu_x l_x=a_0-\frac{c_0}{6}+\frac{e_0}{30}-\frac{g_0}{140}.$$

A full explanation and illustration of the use of this formula will be found in Mr. Sprague's paper "On the value of Annuities payable half-yearly, quarterly," &c., vol. xiii p. 210. By means of it we find that

$$\mu_{30} = \cdot 007613$$
$$\mu_{40} = \cdot 010259$$
$$\mu_{50} = \cdot 015717$$

Also $\delta = \cdot 029558$

$\therefore \qquad \mu_{30.40.50} + \delta = \cdot 063147$ and $10(\mu_{30.40.50} + \delta) = \cdot 63147.$

Thus we get the value of the annuity on the three joint lives,

$$a_{30.40.50} = 11 \cdot 4090 - \cdot 6315$$
$$= 10 \cdot 7775$$

Substituting this value in the place of the less accurate one, 10·7347, employed in the above calculation by the ordinary formula, we get the value of the annuity on the last surviver = 22·6145, which is very nearly the same as that found above, 22·6174, by the direct application of Mr. Woolhouse's formula.

The general result in this instance is, that the values of the annuity on the last surviver of three lives as found by Simpson's formula and by the ordinary one are affected by about the same amount of error, one being in excess and the other in defect, and that the mean of the two is a very close approximation to the true value.

We take this opportunity of pointing out that the still simpler method of finding the value of the annuity by substituting a single life u for the last surviver of y and z, and then finding the value of $a_{\overline{xu}}$, where $a_u = a_{\overline{yz}}$, does not give satisfactory results. Thus, in the above instance $a_{\overline{40.50}} = 19 \cdot 2548 = a_{32}$, and $a_{\overline{30.32}} = 23 \cdot 1224$, which it is seen differs considerably from the true value of $a_{\overline{30.40.50}}$.

We shall be glad to be furnished with the results of any similar comparisons for other ages that may be made by any of our readers. —Ed. *J. I. A.*

HOME AND FOREIGN INTELLIGENCE.

NORTHERN ASSURANCE COMPANY.

Established 1836.

ACTUARIAL REPORT

As to Investigation into the Life Department of the Company's Business for the five years ending 31*st December* 1870.

The amount of New Life Business effected during the five years has been considerably in advance of what was done in the preceding Quinquennium, as the following comparison shows:—

During	No. of New Entrants.	Sums Assured.	Annual Revenue.	Single Premiums.
1861 to 1865.........	2664	£1,445,252	£46,987	
1866 to 1870.........	3447	1,938,306	59,712	£1983
Increase......	783	£493,054	£12,725	£1983

* * * * *

During the Quinquennium 555 deaths have happened among the Lives Assured. The number to be expected by the Carlisle Table was very nearly the same—viz. 558. The Claims by these deaths were for £337,860, in addition to which the sum of £17,301 had to be paid in respect of Endowments, making the total Claims £355,161. The Income for the five years was £700,526. The Interest credited to the Assurance Funds has been at the rate of 4 per cent. The total Expenses of Management (including Commission) have been 8 per cent of the Net Premiums received in the Non-Participating Branch, and 10 per cent in the Participating. For the last two years the Accounts of the Company have been so kept that nothing beyond the commission paid to Agents has been charged in the Non-Participating Branch; the Expenses of Management otherwise having been borne by the Fire Account.

The number of deaths in the Annuity Branch during the Quinquennium has been 48. Nearly the whole of our Annuitants are Females. By the Government Female Annuitants' Mortality (1823) 51 deaths might have been expected. The Expenses of Management (including Commission) have been 5 per cent of the amount paid in the shape of Annuities. The Fund of this Branch also has received interest at the rate of 4 per cent.

The Table which follows shows the Policies in force at 31st December 1870, and the results of the Valuations at that date.

[See pages 272 and 273.]

* * * * *

The following are the BALANCE SHEETS of the different Branches into which the Life Department of the Company's business is divided, as at 31st December 1870.

I.—NON-PARTICIPATING POLICIES.

To Value of £1,016,612 { Annuities 569 } Sums Assured under 1661 existing Policies		£526,221	
Less Value of £159,720 Sums Re-assured		63,550	
			£462,671
„ Surplus			19,906
			£482,577
By Value of £30,948 Annual Premiums (excluding extras)		£378,314	
Deduct Value of Loading		31,950	
		£346,364	
Less Value of £5,064 Annual Premiums (excluding extras) under Re-assurances	£53,124		
Deduct Value of Loading	7,813		
		45,311	
			£301,053
„ Assurance Fund, as per Ledger Account			181,524
			£482,577

II.—PARTICIPATING POLICIES.

To Value of £2,975,517 Sums Assured under 6685 existing Policies		£1,560,719	
Less Value of £104,340 Sums Re-assured		58,490	
			£1,502,229
„ Value of £167,128 Reversionary Bonuses, and £151 Annual Reductions in Premium in lieu of Bonus		£101,977	
Less value of £4,829 Reversionary Bonuses under Re-assurances		3,071	
			98,906
„ Surplus			76,454
			£1,677,589
By Value of £91,305 Annual Premiums (excluding extras)		£1,370,782	
Deduct Value of Loading		257,327	
		£1,113,455	
Less Value of £3,179 Annual Premiums (excluding extras) under Re-assurances	£44,605		
Deduct Value of Loading	4,712		
		39,893	
			£1,073,562
„ Assurance Fund, as per Ledger Account			604,027
			£1,677,589

III.—ANNUITIES.

To Value of 237 existing Annuities for the aggregate amount of £5,530	£54,541
„ Surplus	347
	£54,888
By Sinking Fund, as per Ledger Account	£54,888
	£54,888

ABSTRACT of POLICIES in force at 31st December 1870, and of the VALUATIONS at that date, arranged in Classes.

Classes.	Number of Policies.	Sums Assured.	Reversionary Bonuses and Reductions of Premium.	Office Premiums.	Value of Sums Assured.	Value of Reversionary Bonuses and Reductions of Premium.	Value of Office Premiums.	Value of Loadings.
NON-PARTICIPATING POLICIES.								
		£	£	£	£	£	£	£
No. 1. Single Life. Whole Term. Uniform Premiums	1161	706,371	...	21,735	398,867	...	291,151	23,905
„ 2. Do. do. Increasing do.	146	97,605	...	3,687	54,436	...	50,948	4,194
„ 3. Do. do. Decreasing do.	2	250	...	9	178	...	87	7
„ 4. Do. do. Limited do.	28	13,133	...	53	8,513	...	999	193
„ 5. Endowment Assurances. Uniform Premiums...	60	25,700	...	1,129	18,041	...	9,474	827
„ 6. Do. Increasing do. ...	...	...	...	...	...	...	...	...
„ 7. Joint Life	26	6,470	...	298	4,371	...	3,031	249
„ 8. Last Survivor	13	9,132	...	198	4,359	...	3,325	282
„ 9. Contingent	18	33,785	...	601	7,874	...	5,710	571
„ 10. Short Term	78	101,289	...	2,331	6,761	...	6,793	1,335
„ 11. Endowments	112	22,877	...	788	17,760	...	5,809	290
„ 12. Reversionary and Deferred Annuities	17	Annuities, 569	...	119	1,681	...	987	97
„ 13. Extra Premiums	...	...	...	1,919	3,380	...	...	...
	1661	Annuities, 569 } 1,016,612 }	...	32,867	526,221	...	378,314	31,950
Re-assurances	...	159,720	...	5,202	63,550	...	53,124	7,813
Non-Participating	1661	Annuities, 569 } 856,892 }	...	27,665	462,671	...	325,190	24,137

PARTICIPATING POLICIES.

		£	£	£	£	£	£	£
No. 1. Single Life. Whole Term. Uniform Premiums	5190	2,370,499	Premium Reductions, 131 138,924	71,036	1,230,993	84,193	1,085,267	201,522
„ 2. Do. do. Increasing do.	682	332,180	18 13,761	11,605	170,078	7,751	176,462	32,934
„ 3. Do. do. Decreasing do.	2	1,400	429	18	891	277	214	40
„ 4. Do. do. Limited do.	82	62,791	6,848	844	34,219	4,658	14,692	4,513
„ 5. Endowment Assurances. Uniform Premiums...	587	138,682	1 3,952	5,394	83,626	3,005	65,583	13,034
„ 6. Do. Increasing do. ...	17	5,850	37	246	3,164	25	3,527	699
„ 7. Joint Life	109	32,816	1 2,458	1,370	20,378	1,650	16,946	3,130
„ 8. Last Survivor	8	14,799	719	435	7,782	418	6,345	1,186
„ 9. Contingent	3	3,500	...	80	848	...	671	67
„ 10. Short Term	5	13,000	...	277	1,006	...	1,075	202
„ 11. Endowments	...	...	...	...	...	...	...	...
„ 12. Reversionary and Deferred Annuities	...	...	...	...	...	...	...	...
„ 13. Extra Premiums	...	...	...	4,281	7,734	...	...	...
Re-assurances	6685	2,975,517 104,340	Premium Reductions, 151 167,128 4,829	95,586 3,229	1,560,719 58,490	101,977 3,071	1,370,782 44,605	257,327 4,712
Participating	6685	2,871,177	Premium Reductions, 151 162,299	92,357	1,502,229	98,906	1,326,177	252,615
Total	8346	Annuities, 569 3,728,069	Premium Reductions, 151 162,299	120,022	1,964,900	98,906	1,651,367	276,752

237 Immediate Annuities for the Aggregate Amount of £5530.—Value £54,541.

In the Valuation of Liabilities under existing Policies the Carlisle Table, with interest at 3 per cent, has been employed throughout. Formerly some of the older Policies were valued at 4 per cent, the rate on which their premiums were founded. Under the name "Value of Loading" there has been reserved a sum in excess of the value of the actual Loading, *i.e.* the difference between the Office Premiums and the pure Carlisle 3 per cent Premiums. The increase in the Reserves required by the alterations in the data is £5256 in the Non-Participating Branch, and £12,092 in the Participating, together £17,348.

The surplus in the Participating Branch—£76,454—is sufficient to provide a Reversionary Bonus on all Policies current at 31st December last at the rate of £1:3:1 per cent, upon the sum assured, for every year of their existence since the date of the last division of profits. To provide a Bonus of £1:2:6 per cent would require £74,490, leaving an unappropriated balance of £1964. In addition to showing the above surplus of £76,454, the Company paid £5757 by way of prospective Bonus during the five years now ended.

At the last Investigation the Annuities were valued by the English Life Table, No. 2, at 3 per cent interest, with Loadings varying according to the age of the Annuitant. On this occasion they have been valued by the Government Female Annuitants' Mortality (1823) at 4 per cent interest, with a small uniform Loading. The new data agree far more closely than the old did with the actual circumstances of our Annuity Branch. The Reserve required by the former, however, is much the same—viz. £54,541, while by the latter it would have been £54,761.

[EXTRACT FROM THE REPORT OF THE DIRECTORS.]

The result of the Valuation is that, in each of the three Branches into which the Life Business of the Company divides itself, there is a surplus, viz.:—

1. In the Non-Participation Branch, .	£19,906	0	0
2. In the Participation Branch, of .	76,454	0	0
3. In the Annuity Branch, of . .	347	0	0
Total . . .	£96,707	0	0

As regards the first and third of these (in which the Shareholders alone are interested), the Directors have no proposal to make, further than to recommend that the amounts be left at the credit of the Funds to which they respectively belong.

Participation Branch.

The divisible Surplus upon this Branch, though affected to some extent by the change in the mode of valuing the earlier Policies, will, nevertheless, admit of the payment of a Bonus not greatly below that declared in 1866, and the Directors have pleasure in proposing, in terms of the Actuarial Report, that out of the amount at their disposal, the sum of £74,490 be applied in providing for an addition at the rate of £1:2:6 per cent per annum upon the original amounts insured by all policies current on 31st December last, for the five years ending that date, with the usual option to the insured to receive the present value

of such Bonus in cash, or to apply it in the reduction of their future premiums.

The Directors further recommend that a prospective Bonus, at the same rate, be paid upon all policies which shall become claims before the 31st December, 1875, it being of course understood that, both as regards the immediate and prospective Bonus, the amount is not payable unless the Policy has been five years in existence.

To those to whom it may be a source of disappointment that the former rate of Bonus has not been maintained, the Directors would only say that the additional amount reserved by the present, as compared with previous valuations, has been reserved solely for the benefit of the Policyholders themselves at future Divisions of Profit.

The following further particulars are extracted from the returns under the Fifth and Sixth Schedules of the Life Assurance Companies' Act.

The Life Assurance business of The Northern consists of two distinct branches—the Non-Participation and the Participation.

In both branches the principles on which the Valuation is made are at the discretion of the General Court of Directors.

The whole of the Profits made in the Non-Participation branch belong to the Shareholders.

The 109th Section of the Company's present Act of Incorporation directs as follows respecting the Profits made in the Participation branch:—

> "Subject to such Modification and Regulations as may be "made under the Powers of this Act, with the Sanction and "Approval of the *Committee of Participation Policyholders for "the Time being, the whole Profits of the Participation Branch "shall belong to the Participation Policyholders, under Deduction "of a Sum equal to Ten *per Centum* of the Premiums paid on "that Branch, to be retained by the Company as their Remunera- "tion for managing and carrying on the same, and guaranteeing "the Participation Policyholders against any Deficiency in the "Funds of such Branch, to make good to the Policyholders the "Sums assured by their Policies and declared Bonuses thereon, "such Sum of Ten *per Centum* to include all Charges and "Expenses chargeable against the Participation Branch, with the "Exception of Commissions and Allowances to Agents and others "for introducing Business on the Participation Principle, and "such Profits (after deducting the said Ten *per Centum* and the "Amount of all such Commissions and Allowances) shall be "apportioned by the Company among the Participation Policy- "holders, and applied either in making Additions to the Amounts "assured by the Participation Policies, or, at the option of each "Participation Policyholder, in reducing the future Premiums

* The Participation Policyholders are entitled to appoint Three of their number as a Committee, who have a voice with the General Court of Directors in the investment of the Funds of the Participation Branch, and in the determination of the regulations and practice of the Company with reference to the management of that Branch.

" payable by such Participation Policyholder: Provided always, " that as regards Participation Policyholders assured before the " passing of this Act, nothing in this Section contained shall " authorize the Company to allot to any such Participation " Policyholder a less Share of the Profits actually made in the " Participation Branch than they would have been bound to allot " to him under the Provisions of the hereinbefore recited Act " of 1848."

The Act of 1848 directed as follows:—

" * * * the Profits arising thereon," that is, on the Participation Branch, " shall be applied either in making Additions to the " Policies held by the Parties assured under this System, or in " reducing the annual Premiums payable by them, in their Option, " under the Deduction of Ten per cent on the Premiums, being " the Remuneration agreed to be accepted of by the Company for " managing this Branch of the Business, and guaranteeing those " who prefer assuring under the same, in case the Funds of the " Branch should be found insufficient for that Purpose."

NOTE.—Hitherto the Company has not charged anything against the Participation Branch in addition to ten per cent. on the Premiums paid.

It has been the practice to distribute the Profits among the Policyholders according to a uniform rate of reversionary additions to the sum assured. Such reversionary additions may then, if preferred, be surrendered for their present value in cash, or for a reduction in future Premiums.

The rule has been to reserve the whole of the "Loading;" *i.e.*, the difference between the Premiums payable to the Office and the pure Carlisle Three per cent Premiums. Where the rule has been departed from, a greater amount than the actual "Loading" has been reserved. In the majority of Participation Policies, the "Loading" is $\frac{23}{123}$rds of the Office Premiums. In Non-Participation Policies it is generally $\frac{9}{109}$ths.

A Bonus is set aside for all Participation Policies that are in force at the date of the Valuation, but the same does not become payable until five full years' Premiums have been paid. In the case of Policies which become claims before that time, the original sum assured only is payable, and the value of the Bonus set aside goes to increase the profits at next Investigation.

In the Non-Participation Branch no profit was made, but, after taking credit for the £20,000 allotted to the Branch at the apportionment of the Company's General Reserve Fund consequent on the passing of *The Life Assurance Companies' Act* 1870, there resulted a Surplus of £19,906 0*s.* 7*d.*

The Profit made in the Participation Branch was	£76,453 17*s.* 6*d.*
In the Annuity Branch, . . .	£347 7*s.* 7*d.*

The amount of Profit divided among the Policyholders was, £74,490, and the number of the Policies which participated, was 6,676 for £2,958,814.

REVERSIONARY BONUSES allotted to Policies of £100:—

Age at Entry.	POLICIES IN FORCE FOR						
	5 years.	10 years.	15 years.	20 years.	25 years.	30 years.	35 years.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6
30	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6
40	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6
50	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6	5 12 6

CASH BONUSES *receivable in place of the above Reversionary Bonuses:—*

Age at Entry.	POLICIES IN FORCE FOR						
	5 years.	10 years.	15 years.	20 years.	25 years.	30 years.	35 years.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	1 11 9	1 15 3	1 18 9	2 2 11	2 7 1	2 12 6	2 19 3
30	1 18 9	2 2 11	2 7 1	2 12 6	2 19 3	3 6 4	3 12 2
40	2 7 1	2 12 6	2 19 3	3 6 4	3 12 2	3 19 1	4 5 6
50	2 19 3	3 6 4	3 12 2	3 19 1	4 5 6	4 10 1	4 14 8

REDUCTIONS IN THE ANNUAL PREMIUM *receivable in place of the above Reversionary Bonuses:—*

Age at Entry.	POLICIES IN FORCE FOR						
	5 years.	10 years.	15 years.	20 years.	25 years.	30 years.	35 years.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	0 1 8	0 1 11	0 2 3	0 2 8	0 3 1	0 3 9	0 4 9
30	0 2 3	0 2 8	0 3 1	0 3 9	0 4 9	0 6 2	0 7 9
40	0 3 1	0 3 9	0 4 9	0 6 2	0 7 9	0 10 3	0 13 8
50	0 4 9	0 6 2	0 7 9	0 10 3	0 13 8	0 17 4	1 3 0

Consolidated Revenue Accounts for Five Years, commencing 1st January, 1866, and ending 31st December, 1870.

NON-PARTICIPATION LIFE ACCOUNT.

	£	s.	d.
Amount of Funds on 1st January, 1866, the beginning of the period	130,957	17	2
Premiums (after deduction of Re-Assurance Premiums)	139,356	9	6
Interest	30,752	18	5
*Proportion of General Reserve Fund transferred to this Account	20,000	0	0
Value of Contingent Annuity transferred from Annuity Account	217	16	0
	£321,285	1	1

* This amount was transferred to the Non-Participation Account at the apportionment of the general "Reserve Fund" of the Company consequent on the passing of *The Life Assurance Companies Act* 1870.

	£	s.	d.
Claims under Policies (after deduction of Sums Re-Assured)	120,892	7	5
Surrenders	7,657	19	3
Commission	7,394	11	0
*Expenses of Management	3,816	2	10
Amount of Funds on 31st December, 1870, the end of the period	181,524	0	7
	£321,285	1	1

PARTICIPATION LIFE ACCOUNT.

	£	s.	d.
Amount of Funds on 1st January, 1866, the beginning of the period	394,709	3	8
Premiums (after deduction of Re-Assurance Premiums)	415,284	8	3
Interest	94,914	2	3
	£904,907	14	2

	£	s.	d.
Claims under Policies (after deduction of Sums Re-Assured)	234,268	19	11
Surrenders	25,083	8	3
Commission	24,020	7	1
Expenses of Management	17,508	1	5
Amount of Funds on 31st December, 1870, the end of the period	604,026	17	6
	£904,907	14	2

ANNUITY ACCOUNT.

	£	s.	d.
Amount of Funds on 1st January, 1866, the beginning of the period	53,323	3	5
Sums received as considerations for Annuities sold	19,101	3	3
Interest	10,708	19	11
	£83,133	6	7

	£	s.	d.
Paid to Annuitants	26,692	10	7
Commission	146	2	9
Expenses of Management	1,188	9	8
Value of Contingent Annuity transferred to Non-Participation Life Account	217	16	0
Amount of Funds on 31st December, 1870, the end of the period	54,883	7	7
	£83,133	6	7

The division of the Assurances for Whole Term of Single Life into the first four classes (see Actuarial Report, Abstract of Policies in force) has been made as at the dates when the Policies were taken out. Consequently, a Policy effected by increasing Premiums has been included in the second class, although the Premium, having reached its maximum before the date of the valuation, is now uniform. Similarly, with Policies in the third class. The fourth class includes

* For the Years 1869 and 1870 the Accounts of the Company have been so kept that nothing beyond the Commission paid to Agents has been charged in the Non-Participation Branch; the Expenses of Management otherwise having been borne by the Fire Account.

all Policies which were effected by Single Premiums or by a limited number of Annual Premiums. The Yearly Premiums entered under the second and fourth classes, except where in the former they have reached their maximum, are not those actually current at 31st December last, but the uniform Whole-Life Premiums which, at that date, by Carlisle Three per Cent, have equivalent values. The Office Premiums actually current (*i.e.*, the Premiums payable at the dates of Renewal of the Policies in 1871) are:—(Non-Participating Policies) Class II., £2,696; Class IV., £132. The Premiums under the two Policies in Class III. have reached their minimum. As regards the Participating Policies the same remark applies, with the addition that the Yearly Premiums entered under the Sixth Class are the uniform short term Premiums which are equivalent to the actual increasing ones. The Office Premiums actually current are:—Class II., £9,204; Class IV., 1,877; Class VI., £159. The Premiums under the two Policies in Class III. have reached their minimum.

There are no separate investments for the Life Assurance and Annuity Funds. By arrangement with the "Participation Policy-holders' Committee," the Fund of the Paticipation Branch was credited with Four per Cent interest annually throughout the Quinquennium. The same rate of interest was allowed on the Funds of the Non-Participation and Annuity Branches.

The following is the current practice as to Surrender Values:—Until three full years' Premiums have been paid, no surrender value, as a rule, is allowed. After that, the minimum surrender value allowed for ordinary Whole Life Assurances, and for Endowment Assurances, is thirty per cent of the Premiums paid in the case of Non-Participation Policies, and twenty-five per cent in the case of Participation Policies (in addition to the value of existing Bonuses). In ascertaining the Premiums paid, extras for Foreign Residence and Travel are excluded. Ninety per Cent of the Premiums paid is allowed for Endowments in respect of which it has been stipulated that the whole amount paid shall be returned to the Policyholder in the event of the Life not attaining the specified age. No value, except under special circumstances, is allowed for Endowments effected without the above stipulation as to return of Premiums.

All the business of the Company is considered to be transacted at European Rates. The extra Premiums for foreign residence and travel are arbitrary additions. Policies, carrying such extra Premiums, are taken to form part of the ordinary risks, and are valued on the same principles, except that the amount of the extra Premium does not enter into the calculation, and that, on the other hand, in addition to the ordinary value of the policy, a further sum of about one-third of the whole extra Premium paid is reserved to cover the chance of the Life being deteriorated by the extra risk to which it has been exposed.

In the case of Policies issued at increased Premiums on the lives of persons supposed to have an expectation of life less than the average, the advanced ages for which the Premiums have been charged are taken in all calculations, whether as to the amount to be reserved for the Policy, or the value for surrender.

THE EDINBURGH LIFE ASSURANCE COMPANY.

Founded 1823.

REPORT ON THE INVESTIGATION, AS AT 31ST MARCH 1871, MADE BY THE ACTUARIES OF THE COMPANY TO THE BOARD.

21st June 1871.

The investigation into the affairs of the Company as at 31st March 1871 having now been completed, we beg respectfully to submit to you the following Statement of the Results:—

The Net Liability in respect of Assurance and Annuity Contracts is	£851,848
The Life Assurance and Annuity Funds amount, as per Balance Sheet, to	£978,749
Deducting therefrom the Net Liability as above, . .	851,848
There remains a Cash Surplus of	£126,901
Add Intermediate Bonuses paid since last Investigation in 1863, ...	16,144
Total Profit Realized since last Investigation,	**£143,045**

This sum is divisible as follows, in terms of the Company's Contract:—

To the Assured, nine-tenths of the Total Profit of £143,045, as above . . .		**£128,741 0 0**
Of which there has been already paid by way of Intermediate Bonuses during the currency of the past Septennium, the sum of	£16,144 0 0	
Leaving for distribution now .	112,597 0 0	
	£128,741 0 0	
To the Proprietors, one-tenth of the said Total Profit, ...		**14,304 0 0**
		£143,045 0 0

The sum now available for distribution among the Policyholders will provide Reversionary additions of considerably more than **£180,000.** This, with the Intermediate Bonuses paid during the past Septennium, gives a total of about **£200,000** as the amount of Reversionary additions for the period since last Valuation.

It only remains for us to state the principles upon which the valuation has been made.

The ordinary Assurances and Bonuses have been valued at 3½ per cent. interest, by the "Combined Experience (1863) Table"—a Table requiring a high reserve. The Special Assurances and Bonuses have been valued by the Carlisle 3 per cent. Table, on account of the facilities afforded by the published data on that basis.

In every case in which an extra premium had been charged on account of the life having been considered under the average in point of personal or family history, the Valuation has been made at an increased age corresponding to the premium actually paid. In

valuing the Bonuses, we have made a small additional reserve to meet the deterioration of health of the older lives.

The Annuities have been valued by the Government Annuity Tables at 4 per cent. interest.

For the purpose of comparison, the Ordinary Assurances and Bonuses have been valued by the Table hitherto used by the Company, with special reserves as formerly, and also by the Carlisle 3 per cent. Table, the results in both cases agreeing closely with the Reserve required by the "Combined Experience Table" at $3\frac{1}{2}$ per cent.

Before proceeding to the present Valuation, tests were applied to the previous Valuation of 1863, with the satisfactory result of showing the Reserve then maintained to have been fully in accordance with the results as ascertained by the Table acted on in the present occasion.

Balance-Sheet at 31*st March* 1871.

LIABILITIES.

Paid-up Capital	£75,000	0	0			
Proprietors' Dividend and Reserve Funds	30,610	2	10			
				£105,610	2	10
Assurance Funds				861,237	10	1
Annuity Fund				117,511	14	4
Total Funds as per First Schedule				£1,084,359	7	3
Claims admitted but not paid				26,994	13	0
Sums held in Trust for uncompleted transactions				7,527	9	6
Unpaid Dividends				34	1	5
Expenses due but not yet paid				476	2	1
				£1,119,391	13	3

ASSETS.

Mortgages of Property within the United Kingdom				£540,225	13	9
Mortgages of Property out of the United Kingdom (in Canada only)				20,021	15	7
Loans on Company's Policies (within their Surrender Value)				49,854	10	1
Investments:—						
Colonial Government Securities (Canada Dominion Stock)				32,000	0	0
Railway and other Debentures				224,522	16	5
Railway Preference Shares				26,149	0	0
Company's own Shares (purchased)				9,940	0	0
House Property in Edinburgh, London, Dublin, Glasgow, and Toronto (Freehold and Leasehold)				49,956	15	1
Loans on Assignments of County Rates, etc.				20,808	17	4
Life Interests and Reversions				69,251	1	7
Loans on Personal Security (with Life Policies)				33,866	1	10
Policy Stamps on hand				31	1	6
Cash in Banks:—						
On Deposit	£3,000	0	0			
" Current Account	10,326	0	10			
				£13,326	0	10
Agents' Balances (since paid)				14,517	8	2
Outstanding Premiums				2,064	8	8
Outstanding Interest, due and unpaid at date				1,246	1	9
Do. do. accrued but not yet receivable				11,610	0	8
				£1,119,391	13	3

The following further particulars are extracted from the Returns under the Fifth and Sixth Schedules of the Life Assurance Companies Act:—

The Valuation and the Division of Profits have been made on the following principles:—

(*a*) The Ordinary Whole Term Assurances, and the Immediate Annuities on Single Lives, have been valued in Classes according to age. Other Assurances and Annuities have been valued singly.

In the case of Assurances the age was taken as *the age next birthday* on 31st March 1871; and in the case of Annuities, as *the age last birthday.* Lives assured at rates for an increased age on account of family or personal weakness, have been treated in the Valuation as if the *increased age at entry* had been the *actual age.* In the case of one Annuity an increased age was also assumed.

In cases in which an extra Premium had been charged for Foreign residence, for voyaging, or for employment, a proportion of the extra Premium was set aside to meet the risk for the unexpired period.

The next payments of Premium in the case of Ordinary Assurances were held as due 6, 3, or $1\frac{1}{2}$ months after the date of Valuation, according as the Premiums were payable yearly, half-yearly, or quarterly. All other Assurances were held as having *yearly* Premiums, with the next Premium due six months after the date of Valuation.

The Valuation has been strictly what is called a *Net* or *Pure Premium* Valuation.

The principles upon which the Valuation of Policies is made are not fixed by the Contract of Copartnership, nor by any Regulation or Bye-law of the Company, but are left to the discretion of the Directors—the Contract directing that the Valuation is to be made "according to such data as the Court of Directors shall consider fair and equitable."

(*b*) The Regulation of the allotment of Policy-Holders' share of Profit is vested in the Court of Directors.

The profits have been divided among the Policy-Holders in proportion to the accumulated Net Premiums in each case, under deduction of the value of the Policy. Assurances with Increasing Premiums received half the amount of Bonus during the first term of payment;—all other Assurances on single lives were treated as Ordinary Policies.

In fixing the ratio of participation, Assurances which had already participated were held as if effected at the date of last Valuation at the age then attained. The *Net Premium* and the *Value of Policy* used in fixing the ratio were deduced from the original Tables calculated for the Company, on which the Premiums were based. The sums thus apportioned were converted into additions to the Policies, payable along with the amounts originally assured.

There has not been reserved a fixed proportion of the Annual Premium income to meet expenses and profits, but the whole of the

"Loading," (*i.e.* the difference between the Premiums payable and the net Premiums in each several part of the Valuation, according to the Tables of Mortality and the Rates of Interest respectively employed) has been set aside for this purpose. This Loading averages 21·9 per cent. of the *Office* Premiums, or 28·1 per cent. of the *Net* Premiums in the case of participating Policies, and 11·4 per cent. of the *Office* Premiums, or 12·8 of the *Net* Premiums in the case of Non-Participating Policies.

All Policies on the Profit Scale have had allotted to them a share of the Profits, but Bonuses do not vest until the Policies have been five years in force.

Note.—In future, Policies effected on the ascending scale of Premiums will not participate so long as the first scale of Premium is payable.

The following are specimens of Bonuses allotted to Ordinary Policies for £100, with the amounts apportioned under the various modes in which the Bonus might be received:—

Age at Entry.	NUMBER OF YEARS IN FORCE. 5	10	15	20	25	30	35	40	45	
20	£4 15 0	£7 4 0	£7 5 0	£7 5 0	£7 8 0	£7 10 0	£7 16 0	£8 18 0	£10 16 0	Reversionary Addition.
	1 3 8	1 19 6	2 4 0	2 8 8	2 15 2	3 2 0	3 11 4	4 9 8	5 18 9	Cash Value.
	0 1 1	0 2 0	0 2 4	0 2 10	0 3 5	0 4 3	0 5 7	0 8 1	0 12 9	Reduction of Premium for Life.
30	4 16 0	7 5 0	7 8 0	7 10 0	7 16 0	8 18 0	10 16 0	13 0 0	16 11 0	Reversionary Addition.
	1 9 1	2 8 8	2 15 2	3 2 0	3 11 4	4 9 8	5 18 9	7 14 9	10 10 7	Cash Value.
	0 1 7	0 2 10	0 3 5	0 4 3	0 5 7	0 8 1	0 12 9	1 0 5	1 14 10	Reduction of Premium for Life.
40	4 19 0	7 10 0	7 16 0	8 18 0	10 16 0	13 0 0	16 11 0	22 15 0	..	Reversionary Addition.
	1 16 11	3 2 0	3 11 4	4 9 8	5 18 9	7 14 9	10 10 7	15 4 5	..	Cash Value.
	0 2 3	0 4 3	0 5 7	0 8 1	0 12 9	1 0 5	1 14 10	3 3 1	..	Reduction of Premium for Life.
50	5 8 0	8 18 0	10 16 0	13 0 0	16 11 0	22 15 0	..	..	..	Reversionary Addition.
	2 9 5	4 9 8	5 18 9	7 14 9	10 10 7	15 4 5	..	..	..	Cash Value
	0 3 10	0 8 1	0 12 9	1 0 5	1 14 10	3 3 1	..	..	..	Reduction of Premium for Life.

Consolidated Revenue Account for Seven Years and Seven Months, commencing 1*st September* 1863 *and ending* 31*st March* 1871.

Amount of Funds on 1st September 1863, the beginning of period of Seven years and seven months,		£886,698 3 0
Premiums (after deduction of Re-assurance Premiums),		684,153 18 10
Consideration for Annuities granted,		79,748 2 5
Interest and Dividends,		343,919 3 7
Income-Tax received,		2,521 17 8
		£1,997,041 5 6
Claims under Policies (after deduction of sums Re-assured),		£587,971 11 5
Surrenders,		37,405 0 5
Annuities,		107,681 9 11
Commission,		24,229 8 3
Expenses of Management,		88,439 2 6
Dividends to Shareholders,		54,053 0 1
Income-Tax paid,		6,059 15 1
Loss on Investments,	£13,950 1 1	
Deduct Profit on do.,	7,107 10 6	
		6,842 10 7
Amount of Funds on 31st March 1871, the end of the period,		1,084,359 7 3
		£1,997,041 5 6

* The date of the Annual close of the Books was altered at 1868 from 31st August to 31st March, and the Investigation period has therefore extended on this occasion over *seven years and seven months* instead of *seven years.*

Summary and Valuation of the Policies as at 31st March 1871.

DESCRIPTION OF TRANSACTIONS.	PARTICULARS OF THE POLICIES FOR VALUATION.				VALUATION. Value of Ordinary Policies by the Twenty Offices' Experience Table, Interest 3½ per cent.; of Special Policies and Contingent Annuities by the Carlisle Table, Interest 3 per cent.; and of Annuities other than Contingent by the Government Annuitants' Table, Interest 4 per cent.			
	Number of Policies.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Net Liability.
ASSURANCES.								
I.—WITH PARTICIPATION IN PROFITS.								
For Whole Term of Life,	4,316	£2,677,352 16 1	£74,648 14 6	£57,777	£1,380,112	£996,234	£762,326	£617,786
With Increasing Premiums,	280	173,294 2 0	4,091 0 0	3,264	86,018	89,351	71,477	14,540
With Decreasing Premiums,	10	4,063 7 0	96 4 2	79	2,470	1,054	877	1,594
By Limited Payments,	199	102,726 8 6	3,958 0 8	3,367	47,719	33,801	28,742	18,977
Endowment Assurances,	370	82,745 15 0	3,177 0 11	2,615	46,875	41,080	34,076	12,799
Joint Lives,	75	20,766 0 0	946 3 5	735	12,946	11,487	8,976	3,970
Last Survivor,	12	18,932 8 0	420 12 9	338	10,913	5,865	4,752	6,161
Extra Premiums payable,	..	.. ..	857 8 5	..	429	..	..	429
Total Assurances *With Profits*,	5,262	£3,079,880 16 7*	£88,195 4 10	£68,175	£1,587,482	£1,178,872	£911,226	£676,256
II.—WITHOUT PARTICIPATION IN PROFITS.								
For Whole Term of Life,	824	£580,224 8 4	£19,291 18 5	£17,147	£324,124	£220,038	£194,021	£130,102
With Increasing Premiums,	92	193,371 0 0	4,007 15 2	3,654	95,321	93,864	85,693	9,628
With Decreasing Premiums,	5	1,800 0 0	65 4 10	60	1,121	585	541	580
By Limited Payments,	6	3,783 0 0	194 8 7	177	2,048	599	560	1,489
Endowment Assurances,	12	2,292 8 0	139 12 7	126	1,596	1,305	1,175	421
For Short Terms,	50	35,125 0 0	790 18 1	570	3,555	3,648	3,010	545
Children's Endowments,	18	2,250 0 0	120 5 7	117	1,473	881	862	611
Joint Lives,	8	4,149 19 0	203 14 9	173	2,702	2,192	1,899	803
Last Survivor,	7	4,450 0 0	63 9 7	58	1,615	1,180	1,089	526
Contingent,	16	18,098 19 0	317 4 4	247	4,001	2,276	1,775	2,226
Sum due at fixed Date,	1	800 0 0	2 9 8	2	121	62	61	59
Extra Premiums payable,	..	.. ..	220 11 6	..	119	..	..	119
Total Assurances *Without Profits*,	1,039	£846,344 14 4	£25,417 13 1	£22,331	£437,796	£326,630	£290,686	£147,109
Total Assurances,	6,301	£3,926,225 10 11	£113,612 17 11	£90,506	£2,025,278	£1,505,502	£1,201,912	£823,365
Deduct Re-Assurances,	..	506,930 1 11	13,968 2 6	11,608	263,537	198,848	166,167	97,369
Net Amount of Assurances,	6,301	£3,419,295 9 0	£99,644 15 5	£78,898	£1,761,741	£1,306,654	£1,035,745	£725,996
ANNUITIES.		(PER ANNUM.)						
Immediate—on Single Lives,	272	£15,080 4 3	..	..	£122,702	..	..	£122,702
Ditto Joint Lives,	1	55 1 0	..	..	240	..	..	239
Ditto Last Survivor,	3	66 0 6	..	..	940	..	..	940
Ditto Annuity Certain,	1	34 19 0	..	..	244	..	..	244
Deferred,	5	197 11 6	..	..	1,572	..	..	1,572
Contingent,	5	1,370 0 0	£303 16 7	£218	2,093	£2,600	£1,853	240
Total Annuities,	287	£16,803 16 3	£303 16 7	£218	£127,791	£2,600	£1,853	£125,937
Deduct Contingent Annuity Re-Assured,		500 0 0	118 7 6	84	817	1,034	732	85
Net Amount of Annuities,	287	£16,303 16 3	£185 9 1	£134	£126,974	£1,566	£1,121	£125,852
Total of the Results,			£99,830 4 6	£79,032	£1,888,715	£1,308,220	£1,036,866	£851,848

* Sum Assured, £2,892,396. 16s. 7d.; Bonuses, £187,484.

Valuation Balance-Sheet, as at 31st March 1871.

Dr.

	£	s.	d.
To Net Liability under Assurance and Annuity Transactions (as per Summary Statement), ...	£851,848	7	8
To Surplus, ...	126,900	16	9
	£978,749	4	5

Cr.

	£	s.	d.
By Life Assurance and Annuity Funds (as per Balance-Sheet), ...	£978,749	4	5
	£978,749	4	5

The average rate of interest at which the Life Assurance Fund of the Company was invested at the close of each year during the period since last Investigation was as follows:—

Date.	Rate of Interest.	Date.	Rate of Interest.
1864, Aug. 31st.	£4 13 5	1868, March 31st.	£4 7 3
1865, Aug. 31st.	4 12 5	1869, March 31st.	4 6 5
1866, Aug. 31st.	4 10 4	1870, March 31st.	4 7 11
1867, Aug. 31st.	4 8 1	1871, March 31st.	4 8 1

*** The period of the Annual close of the Books was altered at 1868 from 31st August to 31st March.

The Company has since last Investigation allowed for the Surrender of ordinary Whole Term Assurances, in the case of Participating Policies, 75 per cent., and in the case of Non-Participating Policies 67½ per cent. of the net values of the Policies deduced from the Table used in the formation of the premiums.

The following are Specimens of the Surrender Values which have been allowed for ordinary Participating Policies, exclusive of Bonus Additions.

Age at date of Assurance.	Surrender Values of Policy for £100.			
	Duration 5 years.	Duration 10 years.	Duration 15 years.	Duration 20 years.
20	£2 11 0	£5 15 0	£9 5 0	£13 3 0
30	3 11 0	7 16 0	12 11 0	18 0 0
40	5 1 0	11 3 0	17 17 0	24 14 0
50	7 13 0	15 17 0	23 16 0	32 6 0

The following Table shows the Cash Values which have been allowed for the Surrender of Bonus Additions in the case of ordinary Policies.

Age next Birthday.	Cash Value allowed for each £10 of Bonus.	Age next Birthday.	Cash Value allowed for each £10 of Bonus.	Age next Birthday.	Cash Value allowed for each £10 of Bonus.
25	£2 9 10	50	£4 2 8	75	£6 7 3
30	2 14 11	55	4 11 6	80	6 13 10
35	3 0 8	60	5 0 9	85	6 19 7
40	3 7 2	65	5 10 0	90	7 3 10
45	3 14 7	70	5 19 1		

The Company has never been asked to give a surrender value for an Endowment. In the case of Endowment Assurances the value allowed has never been less than 75 per cent. in the case of Participating Policies, and 67½ per cent. in the case of Non-Participating Policies, of the net value deduced from the Table used in the formation of the premiums, but in many instances larger values have been allowed.

Policies on unhealthy lives have been valued at the Investigation at the advanced ages corresponding to the rate of premium charged, and in the Return under Heading No. 2, they are dealt with in the same manner. In valuing such Policies for surrender however the actual ages are taken.

WESTMINSTER AND GENERAL LIFE ASSURANCE ASSOCIATION.

Established 1836.

REPORT OF THE DIRECTORS.

On the occasion of the Seventh Quinquennial General Meeting, the Directors present to the Shareholders and Assured a Report of the progress of the Association during the last five years, with a statement of its present position.

During the last quinquennium the Association has issued 1639 Policies, assuring £596,324., and yielding a Premium Income of £19,145. 5*s*. 1*d*., being an increase over the new business transacted during the previous five years of 739 Policies, £232,377., Sums Assured, and £6,838. 14*s*. 3*d*. in Premiums.

In the same period the yearly Income has increased by £12,065., the Receipts for the year 1871 being £53,683.

The Assets have increased £81,088. The total Funds of the Association being on the 31st December last £338,669. 4*s*. 9*d*., viz.:—

Guarantee Fund	£13,866	9	10
Assurance Fund	324,802	14	11
	£338,669	**4**	**9**

In addition to a Balance of Subscribed Capital not called up of **£95,000.**

During the past five years 203 Policies effected on 181 lives have become Claims, on which, with bonus additions, the sum of £106,996. 1*s*. 8*d*. has been paid or provided for.

The Valuation of the Assets and Liabilities appended to this report is based on the Carlisle table of mortality, it being assumed that the Funds of the Association will be improved at 3 per cent. interest. From the present values of the premiums receivable the sum of £106,288. 13*s*. 9*d*. has been deducted to provide for future expenses of management and profit. The amount of Assets over Liabilities shewn is therefore realized profit, no portion of profit being anticipated.

The Assets of the Association have been according to the provisions of the Deed of Settlement valued at cost price.

Adopting these methods of valuation, shewing a reserve of £280,352. 17*s*. 3*d*. in addition to the premiums hereafter to be

received, to provide for the payment of claims on existing Policies, the Directors are gratified in announcing that the result of the present investigation shews the profit on the business of the last five years to be £36,212. 5*s.* 6*d.*

By the Deed of Settlement of the Association it is provided that one-tenth thereof or £3,621. 4*s.* 6*d.* shall be added to the Guarantee Fund, which will with such increase stand at £17,487. 14*s.* 4*d.*

One-tenth or £3,621. 4*s.* 6*d.* to be divided among the Shareholders, and Four-fifths or £28,969. 16*s.* 6*d.* will be divided among the Assured entitled to participate therein, which will give a cash Bonus of about 20 per cent. on the Premiums paid since they last participated, or a Reversionary addition to the sums Assured by their Policies varying according to age from 50 to 38 per cent. on the amounts so paid.

Balance Sheet of The Westminster and General Life Assurance Association on the 31*st December*, 1871.

LIABILITIES.

				£	s.	d.
Shareholders' Capital paid up and additions	£13,866	9	10			
Interest thereon payable to Shareholders for Dividend, 1871	534	11	10			
Assurance Fund	316,565	2	9			
Redemption Fund (28, King Street, Covent Garden)	346	9	2			
Reserve Fund (Loans on Personal Security)	234	17	8			
Total Funds, as per First Schedule				331,547	11	3
Claims admitted but not paid				6,623	6	10
Unpaid Accounts				342	18	4
„ Ground Rent				21	10	0
„ Cash Bonuses				133	18	4
				£338,669	4	9

ASSETS.

				£	s.	d.
Mortgages on Property within the United Kingdom				46,870	0	0
Loans on the Company's Policies				18,776	11	8
Investments—						
In British Government Securities				50,716	4	4
Indian and Colonial Government Securities				42,122	15	0
Railway and other Debentures and Debenture Stocks				140,870	4	4
House Property (Offices, 28, King Street)				4,218	15	5
Parochial Rates				5,700	0	0
Life Interests				3,992	8	4
Reversions				400	0	0
Office Furniture				500	0	0
Loans upon Personal Security				3,604	3	4
Agents' Balances				2,669	1	3
Outstanding Premiums				5,663	18	5
Ditto Interest				4,458	6	1
Cash—						
On Deposit	£4,000	0	0			
In hand and on Current Account	929	7	7			
				4,929	7	7
Value of Re-assurance Policies				3,151	0	0
Rent of Chambers due				26	9	0
				£338,669	4	9

The following further particulars are extracted from the Returns under the Fifth and Sixth Schedules of the Life Assurance Companies Act:—

The principles of valuation are determined by the Deed of Settlement of the Association.

Assurances for the whole term of life are classified according to age which will be attained by the life assured on the birthday following the date on which the renewal premium on the policy becomes due, and the present values of the sums assured by the policies and the premiums expected to be received on them ascertained. The values of the premiums (less the values of the additions made to them for expenses of management and future profits) are deducted from the values of the sums assured, and the difference taken as the liability of the Association in respect of the Assurances. The Endowment Assurances, payable at age 60, are similarly classified and valued; all other Assurances are valued separately on the same principle. The Immediate Annuities on single lives for the whole term are classified according to the age of the annuitant, and the value of the Annuity at each age ascertained; other Annuities are valued separately.

The Assured entitled to participate, share in the amount divisible amongst them, in proportion to the Premiums they have paid since they last participated, improved at 3 per cent. interest, previous additions to their Policies being treated as new Assurances effected at the date of such addition.

The whole of the Loading has been reserved as a provision for future expenses and profits.

The Consolidated Revenue Account of The Westminster and General Life Assurance Association, for Five Years, commencing 1*st January*, 1867, *and ending* 31*st December*, 1871.

	£	s.	d.
Amount of Funds on 31st December, 1866	257,580	7	2
Premiums after deduction of Re-assurance Premiums	176,659	3	7
Consideration for Annuities granted	10,645	4	9
Interest and Dividends	57,950	3	0
Fines	42	0	0
Rent of Chambers	562	14	0
Sundries	497	5	3
Increase Value of Re-assurance Policies	1,551	0	0
Unpaid Bonuses on Lapsed Policies	5	6	2
	£505,493	3	11

	£	s.	d.
Claims under Policies (after deduction of sums reassured)	106,996	1	8
Surrenders	6,117	13	6
Annuities	12,660	13	1
Commission	9,089	18	9
Expenses of Management	23,471	18	0
Dividends and Bonuses to Shareholders	5,309	6	6
Cash Bonuses to Assured	9,555	16	8
Agents' Balances written off	199	0	11
Repairs, Office Fittings and Furniture	351	19	4
Interest on Loans, Purchases, &c.	193	4	3
	173,945	12	8
Amount of Funds at the end of the Year 1871	331,547	11	3
	£505,493	3	11

Summary and Valuation of Policies of the Westminster and General Life Assurance Association, as at 31st December, 187[illegible].

DESCRIPTION OF TRANSACTIONS.	PARTICULARS OF THE POLICIES FOR VALUATION.				VALUATION. Value by the Carlisle Table—Interest 3 per Cent.			
	Number of Policies.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums if ascertained.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Net Liability.
ASSURANCES.								
I.—*With Participation in Profits.*		£.	£.		£.	£.		£.
For the whole term of Life	2489	1061715·471	31116·562		573418·176	453525·335		210597·908
Ascending Premiums	5	2100·	24·579		840·898	1016·214		27·928
Limited Premiums	12	6816·690	88·719		5328·234	270·280		5112·010
Joint Lives	108	24434·230	1163·954		14874·132	13947·768		3715·918
Longest of Two or more Lives	7	9309·060	149·771		4076·711	2663·657		1945·785
Extra Premiums payable			260·925					440·692
Total Assurances with Profits	2621	1104375·451	32804·510		598538·151	471423·254		221840·241
II.—*Without Participation in Profits.*								
For the whole term of Life	248	116943·	4010·425		64141·747	55865·310		16013·783
Ascending Premiums	1	2000·	29·500		897·340	930·606		95·623
Limited Premiums	4	1550·			678·854			678·854
Joint Lives	12	2250·	105·922		1426·899	1225·522		371·112
Longest of Two or more Lives	3	3400·	69·100		1607·732	1178·672		592·306
Endowment Assurances	207	52750·	1973·029		34675·780	20376·776		17121·188
Endowments	42	6844·9	269·558		4483·765	1995·455		2764·681
Survivorships	13	9300·	224·374		1704·040	1549·107		369·484
Survivorship and Issue	1	3000·			75·			75·
Issue	3	4100·			439·500			439·500
Terms of Years	30	14000·	207·588		922·168	819·808		215·902
Extra Premiums payable			10·500					
Total Assurances without Profits	564	216137·9	6899·996		111052·825	83941·256		38737·433
Total Assurances	3185	1320513·351	39704·506		709590·976	555364·510		260577·674
ANNUITIES.								
Immediate	64				19616·525			19616·525
Contingent	1		17·550		191·700	136·540		74·070
Deferred	1		8·487		110·357	29·907		84·592
Total Results	3251	1320513·351	39730·543		729509·558	555530·957		280352·861
Re-assurances	21	32000·	1017·910	Valued at £3151·				

Valuation Balance Sheet of the Westminster and General Life Assurance Association, as at 31st December, 1871.

DR.	£	s.	d.
To Net Liability under Assurance and Annuity Transactions as per Summary Statement	280,352	17	3
To Surplus	36,212	5	6
	£316,565	2	9

CR.	£	s.	d.
By Life Assurance Fund as per Balance Sheet	316,565	2	9
	£316,565	2	9

Two Years' Premiums are required to have been paid on a Policy before it is entitled to participate in the profits.

£28,969. 16*s*. 6*d*. was divided amongst the Policy holders. The number of Policies which participated was 2,322; and the sums assured thereby £974,559.

Specimens of Bonuses allotted to Policies for £100., effected at the under-mentioned ages, according to the various modes in which the Bonus might be received.

POLICY IN FORCE.

Office Age at Entry.	Five Years. Cash.	Five Years. Reversionary Addition.	Five Years. Reduction of Annual Premium.	Ten Years. Cash.	Ten Years. Reversionary Addition.	Ten Years. Reduction of Annual Premium.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	1 14 7	4 10 0	0 1 10	1 14 7	4 4 0	0 1 11
30	2 5 2	5 2 0	0 2 8	2 5 2	4 15 0	0 2 10
40	3 0 2	5 18 0	0 4 1	3 0 2	5 9 0	0 4 7
50	4 3 10	7 0 0	0 7 1	4 3 10	6 10 0	0 8 2

POLICY IN FORCE.

Office Age at Entry.	Fifteen Years. Cash.	Fifteen Years. Reversionary Addition.	Fifteen Years. Reduction of Annual Premium.	Twenty Years. Cash.	Twenty Years. Reversionary Addition.	Twenty Years. Reduction of Annual Premium.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	1 14 7	3 18 0	0 2 0	1 14 7	3 12 0	0 2 2
30	2 5 2	4 9 0	0 3 1	2 5 2	4 2 0	0 3 5
40	3 0 2	5 1 0	0 5 1	3 0 2	4 13 0	0 5 10
50	4 3 10	6 3 0	0 9 4	4 3 10	5 16 0	0 11 3

On 31st Dec., 1867, the Investments yielded £4 0 1 per cent.
" 1868, " " £3 18 10 "
" 1869, " " £3 19 0 "
" 1870, " " £4 3 10 "
" 1871, " " £4 7 3 "

The average rate of Interest yielded during the five years by the whole of the Funds, whether invested or uninvested, was £3. 18. 0. per cent. per annum.

A Table of Minimum Values allowed for the surrender of Policies for the whole term of Life, and for Endowments and Endowment Assurances.

Sum Assured, £100.

Age at Commencement of the Policy.	TERM IN FORCE.							
	Five Years.		Ten Years		Fifteen Years.		Twenty Years.	
	Participating.	Non Participating.	Participating.	Non Participating.	Participating.	Non Participating.	Participating.	Non Participating.
	£. s. d.	£. s. d.	£. s. d.	£. s. d.	£. s. d.	£. s. d.	£. s. d.	£. s. d.
20	3 8 0	3 0 0	7 0 0	6 4 0	10 14 0	9 10 0	15 0 0	13 6 0
25	3 16 0	3 8 0	7 14 0	6 16 0	12 2 0	10 16 0	16 12 0	14 14 0
30	4 0 0	3 12 0	8 14 0	7 16 0	13 8 0	11 18 0	19 2 0	17 0 0
35	4 18 0	4 8 0	9 18 0	8 16 0	15 19 0	14 2 0	23 5 0	20 12 0
40	5 4 0	4 14 0	11 14 0	10 8 0	19 10 0	17 8 0	27 8 0	24 8 0
45	6 18 0	6 2 0	15 6 0	13 12 0	23 18 0	21 4 0	30 18 0	27 8 0
50	9 6 0	8 4 0	18 14 0	16 12 0	26 8 0	23 8 0	35 4 0	31 4 0

To the above sums must be added the present Values of Reversionary Bonuses allotted to Participating Policies.

The Sum Allowed on Surrender of Endowment or Endowment Assurance Policies (after the receipt of Two Annual Premiums thereon) is not less than One Half of the Premiums paid on them.

The Association does not transact business at other than European Rates.

When an increased rate of Premium is charged for an Assurance, in consequence of the expectation of the life being estimated by the Medical Officer as below the average, the Policy is valued as though the age corresponding to the increased rate had been the real age of the life Assured when the Policy was effected.

NOTICES OF NEW BOOKS.

Life Insurance in 1872; *being a Summary and Analysis of the Accounts of the Life Insurance Companies of Great Britain and Ireland, as now for the first time exhibited by the Returns deposited with the Board of Trade in pursuance of the Life Assurance Companies Act,* 1870. By T. B. Sprague, M.A., Vice-President of the Institute of Actuaries. Part I. London: Charles and Edwin Layton, Fleet Street.

Instead of giving an original notice of this work, we lay before our readers the translation of an article contributed to the *Moniteur des Assurances* by M. V. Senés, which, while giving for the information of the public in France a review of the pamphlet before us, adds incidentally some interesting comparisons between the English and French

companies, and furnishes a good deal of information regarding the manner in which the latter conduct their business. For these reasons, the article, we believe, cannot fail to interest the readers of the *Journal.*

This work contains an analysis and summary of the accounts of the English Life Insurance Companies, and is prefaced by an interesting introduction upon the antecedents of the law. The author promises a second part, in which the materials here furnished will be applied to exhibit the relative position of the various companies, and enable the reader to make comparisons between them in various respects.

The first part of this work deserves a careful study. It contains interesting facts as to the past and present of life insurance in England, which our (French) readers will be glad to have laid before them. It will also enable us to give an exact comparison of the present situation of France and England with regard to life insurance.

Prior to the passing of the Life Assurance Companies Act, 1870, the publication of accounts was not so general as it should have been. Indeed, the affairs of many of the older companies were involved in the greatest secresy. They asked for the confidence and support of the public simply on the ground of the high standing of their Directors. The Joint Stock Companies Act of 1844 gave birth to a crowd of life insurance companies. Altho' by that Act the companies were required to file their accounts, no particular form of account was specified, and no penalty attached to the omission to file them. In consequence of the reckless competition between these companies, the greater number found themselves, after a brief career, unable to continue their business. The grave question then arose, how was the business of these companies to be wound up.

The author justly remarks that the business of a life insurance company differs from that of other companies in this respect, that in consequence of the inevitable deterioration in the state of health of many of the lives assured after the lapse of a number of years, it is not possible, when the company wishes to retire from business, to assess with any approach to equity the interest of each policyholder in the assets, so as to pay him his share in cash. The only proper method of providing for the claims of the policyholders is to obtain the guarantee of some solvent company that the contracts shall be duly fulfilled. This is what has been called, with more or less propriety, an amalgamation. Unhappily, transfers of this kind have given rise to innumerable frauds. The shareholders, and still more, the negotiaters who arranged the terms of the transfers, studied their own interests without troubling themselves much about the fate of the assured. On the other hand, the companies which recklessly undertook the business of others in order to swell the total of their transactions, were hastening to inevitable ruin. Two of the companies which transacted the greatest number of amalgamations, the Albert and the European, have acquired a discreditable notoriety; and their failure brought life insurance into some discredit in England for the time.

The insolvency of these two companies has led men to consider how similar scandals can be prevented in future. Several systems have been discussed: should the State, it has been asked, have a monopoly of life insurance business? or should it exercise the strict supervision over

the companies which is practised in the United States? Both these systems, being considered too despotic, and inconsistent with the principles of free trade, so dear to the English, have been wisely rejected. A compromise has been adopted, which will allow the public to make enquiries and form their own opinion as to the solvency of insurance companies. The Act of 1870, it must be confessed, is not perfect. It was amended in 1871, and again this year, and it will have to be amended hereafter in many particulars; but in our opinion it is a step in the right direction. The nature of the business of life insurance companies justifies a certain amount of legislative intervention. When this intervention is limited to the issue of regulations as to the founding of new companies and as to the publication of annual accounts in a prescribed form, thus enabling the persons interested, or experienced actuaries, to examine and discuss such accounts, legislation does not overstep its proper limits. In France, the legislature has so understood its functions, at least as regards providing for the founding of companies; and if no particular form of annual account has been prescribed, it must be admitted that all the French companies invite investigation by the completeness of their accounts, and the explanations which accompany them.

Mr. Sprague has well indicated the advantages which the English companies will derive from the publication of their accounts; public confidence cannot but increase and be strengthened in consequence. Every item in the accounts will be narrowly scrutinized and examined by the managers; and one result of the new Act will certainly be a reduction in the general expenses. We may add, that the comparison of all the companies, facilitated by the system of uniform accounts, will be the real stimulant of competition and a check on their management. Companies are no longer allowed to put forward such points as are favourable to them, and suppress others which might injure their credit: they must all publish the same particulars; and the English press, so watchful of the interests of the public, will know how to obtain from the Act all the advantages it admits of. The Author sets an example which will certainly be followed by many others.

Hitherto, strange to say, there has been no means of ascertaining, even approximately, the extent of the business and the assets of the English life insurance companies. The Act of 1870 furnishes for the first time information on various important points, such as the annual receipts for premiums and interest, the total assets, and the yearly payments for claims. The accounts filed do not give any direct information as to the total sum assured or the number of the policies in each company—an omission which will probably soon be supplied by an amendment of the Act. The accounts also give no information as to the number and amount of the new insurances; and this again is an omission which ought to be supplied.

One hundred and eight companies have filed their accounts under the Act, but among these are included two American companies having agencies in England, and the Law Reversionary Interest Society, which is in no sense a life insurance company. These three the Author has omitted from his list, stating as his reasons for omitting the former that they have probably as yet transacted very little business in England, but still more that he wished to give a view of the position of the British

companies alone. He has added to the accounts contained in the blue-book those of the following companies, which had been omitted for various reasons:—the British Imperial, Colonial, Imperial Union, Norwich Provident, and Scottish Union; and his tables therefore relate to 110 companies. We learn from other sources that there are 120 companies in England. Several of the less important of these did not deposit their accounts before 31 December 1871, not being required to to do so until nine months after they have been presented to the general meeting of the shareholders.*

Mr. Sprague has drawn up, from the published accounts, four tables, of which the first and fourth, in particular, afford much valuable information. The first shows the income and outgo of 110 companies, and may be summarized as follows:—

INCOME.

	£
Premiums	9,935,165
Consideration for Annuities	258,656
Interest and Dividends	4,090,437
Sundries	93,084
Total	£14,377,342

OUTGO.

	£
Claims	8,290,894
Surrenders	559,604
Annuities	417,010
Commission	410,436
Expenses of Management	1,003,898
Dividends and Bonuses to Shareholders	507,743
Bonus in Cash, and reduction of Premium	509,883
Sundries	68,001
Total	£11,767,469

This table speaks for itself. The first item of the receipts shows us that the premiums paid to the companies in the year 1871† were nearly £10,000,000. Assuming an average premium of £3 per-cent, we get a total sum assured of about £331,172,167. A reference to the accounts of the French companies and M. Ch. Levasseur's article in the last number of the *Moniteur des Assurances*, entitled "Position and progress of Life Insurance in France," will show to what an enormous extent England is ahead of France in the matter of life insurance. The insurances in force with the French companies on 31 December 1871 amounted only to £38,920,000. This is certainly a large sum; but it falls far short of the sum insured in England; and yet France is eminently a saving country, and its population exceeds that of England. This inferiority arises solely from the ignorance of the masses as to the benefits of life insurance; and everything that tends to dissipate that ignorance, whether it be the competition of the

* It is scarcely necessary to point out that our contemporary is here slightly inaccurate; as the Act makes no reference whatever to the presentation of the accounts to the shareholders, and the nine months are reckoned from the day to which the accounts are made up.—ED. *J. I. A.*

† This is also a slight inaccuracy: the year to which the accounts relate being, in most cases, 1870.—ED. *J. I. A.*

companies, old or new, or the advocacy of the press and of books, or the ceaseless exertions of the agents, should be gratefully acknowledged. France is nearly £300,000,000 in arrear of England!

The other figures in the table prove still more how much we are behindhand. The English companies paid in the year to their assured £8,290,894; but the French companies only paid £586,640. Eight millions sterling coming like a gift from heaven to the widow, the orphan, and the destitute! Truly, a splendid result!

The foregoing figures suggest some comparisons:—

The English companies have paid in the year £8,290,894 out of £331,172,167 assured; that is about 2½ per-cent. The French companies have paid £586,640 out of £38,920,000, or about 1½ per-cent only. The same remark presents itself under another aspect. In England, £8,290,894 has been paid to the assured, and the offices have received £9,935,165. The claims are therefore 83 per-cent of the premiums. In France, where £1,440,000 has been received in premiums, the claims paid have been £586,640 only, or very nearly 40 per-cent. It may be supposed by some that these results can be explained by the well known prudence of the French companies; but other causes explain, not only the difference between the above ratios, but also the increase in the death rate experienced by the French companies for some years past.

M. Levasseur has pointed out that the claims, which in 1871 amounted to £586,640, were only £332,520 in 1869, and £385,720 in 1870. The reports of the companies attribute this increase solely to the influence of the war and the epidemic of small-pox, which have lately ravaged France. These have certainly caused the greatest part of the increase; but there is one very important cause which it will be well to point out for the purpose of explaining how it comes that the percentages of the claims to the sums assured in France and England should differ to such an extent as we have just shown.

Some of the French life insurance companies are very old, if we regard the date of their establishment; but they are all what the English call young offices, if we take into account the slow progress they formerly made. It is generally known that the French life insurance companies made little progress for many years during the baleful prevalence of the tontine system, which nearly all of them encouraged to their own detriment, and that it is only about twelve years ago that their life insurance business proper can be said to have really begun to increase.

This is proved beyond all doubt by statistics. M. Levasseur's table, in the above quoted article, shows us that out of the total of £69,872,000 sums assured by the French companies from 1819 to 1871, only £13,360,000 was assured between 1819 and 1859. The prosperity and real activity of the French companies date from this latter year. This is now indisputable, the causes and promoters of this prosperity being well known. Now English actuaries have given particular attention to the increase in the ratio of the claims to the premiums. The premiums, which in the earlier years are more than sufficient to pay the claims, at last become insufficient, the difference being supplied by means of the reserves so carefully accumulated by the companies.

Returning to Mr. Sprague's table, we notice other points of interest. Thus, the sums paid for surrenders were £559,604. The expenses of management amount to £1,003,898, which is a little more than seven per-cent of the total income. The dividends paid to the shareholders and those paid to policyholders are equal in amount, and are together about seven per-cent of the premiums and interest received by the companies. Thus it appears that the English companies, while professing to give the assured a much higher share of the profits, in reality only give them the same proportion as the French companies,—one-half.* This is a fact which is well worthy of notice.

Has the business of the English companies increased of late years? Although this appears probable, the published accounts throw no light on the question. The comparison of the next year's accounts with those of the present will answer it, and meantime it is useless to indulge in conjectures which have no solid basis. The answer, as regards France, is unhappily in the negative; the causes of this temporary stagnation being too well known to require mention. We are convinced that the cause having ceased, the effect has ceased also, and that the French companies will resume the career of prosperity which has been for a time interrupted.

Mr. Sprague's fourth table shows the nature and amount of the investments of the 110 companies. It appears to us of sufficient interest to be reproduced in full.

	£
Mortgages on Property within the United Kingdom	61,013,570
„ „ out of „ „	751,482
Loans on the company's Policies	5,269,363
British Government Securities	8,214,559
Indian and Colonial do.	5,255,758
Foreign do.	1,228,494
Railway and other Debentures and Debenture Stocks	10,412,626
Railway and other Shares, Preference and Ordinary	2,730,245
Landed and House Property	4,691,432
Reversions and Life Interests	1,689,322
Company's own Shares	381,032
Loans on Personal Security	1,806,924
Agents' Balances	1,911,055
Outstanding Premiums	485,081
„ Interest	776,763
Cash on Deposit	1,398,842
„ in hand	1,310,171
Other Assets	283,521
Total	£109,610,235

* Our contemporary has here fallen into a rather surprising misconception; overlooking the circumstance that by far the greatest part of the dividends and bonuses to shareholders, £507,743, consists of interest on their capital, which is, of course, no part of the *profit* of the companies, but is actually yielded by the mortgages or other securities on which the capital is invested. The total funds of the companies at the beginning of the year being £90,846,770, and the interest received in the year £4,090,437, it follows that the average rate of interest is almost exactly 4½ per-cent. The shareholders' capital, £10,376,321, would at this rate produce a yearly interest of £466,935, and it is only the excess over this, or £40,808, which is the shareholders' proportion of profits; and this is only 8 per-cent of the amount appearing in the table as cash bonuses to policyholders, which, it is to be observed, is quite independent of the probably much larger amount of bonus applied to increase the sum assured.—ED. *J. I. A.*

The investments of the English companies amount to nearly 110 millions sterling. An examination of this table brings to light some remarkable differences between the investments of the English companies and those of the French. The principal part of these investments, representing nearly 60 per-cent of the assets, consists of mortgages,—a method of investment which is but little used in France. On the other hand, the British Government securities amount only to about 12 per-cent of the assets; whereas in France government funds constitute the chief investment. This arises from the difference in the rate of interest. The English funds produce only 3 to 3½ per-cent, while the French, under ordinary circumstances, produce from 4½ to 5 per-cent. The companies naturally look for the highest rate of interest that can be obtained with complete security. Thus it is that the investments in landed and house property amount only to £4,691,432, or 4 per-cent of the assets, whereas in the French companies, particularly in the older ones, they form one of the principal items of the assets.

Lastly, comparing the total assets, £109,610,235, with the interest shown in the first table, we see that the English companies, which calculate their premiums at 3 or 3½ per-cent interest, obtain a return on their investments of about 4 per-cent.* The French companies, which compute their premiums at 4 per-cent interest, realize from 5 to 5½ per-cent on their investments.

We will not carry these comparisons any further. Mr. Sprague's work, compiled with his usual care and ability, has furnished us with valuable information as to the position of life insurance in England. We will lose no time in noticing the second part of that work as soon as it appears. It seems to us that the conclusion to be drawn from the above statistical summary is very simple:—the French life insurance companies have certainly much to do to bring their business to the English level; and yet in France the field is more extended, and there are more insurable lives. Nothing is required but vigorous and sustained efforts, if not to reach the goal, at least to approach nearer and nearer to it. The French companies, whether old or new, should spare no pains to accomplish this work, and to sustain the recent prosperity of the institution. May they never forget, while endeavouring to extend their business, that their own interest is identical with the general interest of the country.

* The comparison here made is not strictly correct. The total assets of £109,610,235, include not only the share capital and life funds at the end of the year, £93,456,643, as given in Table I, but also the fire and other funds of £12,278,861, as given in Table II; in addition to other sums, such as claims admitted but not paid, &c. The interest to be compared with this total should be, therefore, the sum of those given in Tables I and II, £4,090,437 and £522,686, together £4,613,123; and the average rate of interest is then found to be £4. 4*s*. 2*d*. per-cent. But, in our opinion, the more correct course will be to compare the total "funds at the *beginning* of the year," £90,846,770, as given in Table I, and £11,766,690, as given in Table II, together £102,613,460, with the total interest, £4,613,123; and this raises the average rate of interest to £4. 9*s*. 11*d*. per-cent. —Ed. *J. I. A.*

CORRESPONDENCE.

MR. SANG'S SEVEN-FIGURE LOGARITHMS.

In the July Number of the *Journal* we inserted a letter from Mr. Sang communicating an error in his new table of seven-place logarithms. A similar communication made by Mr. Sang to the *Athenæum* about the same time drew forth three letters, which we here reproduce together with Mr. Sang's reply. It will be noticed that Mr. Glaisher points out another error in Mr. Sang's Table, namely in log 38962.

LOG OF 52943.

Vega's ten-place logarithms (Leipzig, 1794), which follow Vlacq's, point out Vlacq's error in this and ninety-nine other logarithms. It is remarkable that though Vega gives the log correctly as 7238085468, Hülsse's abridgment of Vega (Leipzig, 1840) has the error 7238086. But the error has not escaped detection, for Schrön's tables (Braunschweig, 1861) give correctly 7238085. Seeing that, according to Mr. Sang, Callet, Sherwin, Hutton, Babbage, Taylor, Shortrede, John Newton (to which we may add W. Gardiner, 1742), give Vlacq's error in this case, it might be worth while verifying the other ninety-nine cases mentioned by Vega.

ALEXANDER J. ELLIS.

TRINITY COLLEGE, CAMBRIDGE,
10 *June* 1872.

In the current number of the *Athenæum*, Mr. Sang publishes an error in log 52943 in his table which was communicated to him by Dr. A. W. Whitcom, of Milwaukee, U.S. The error occurs in Vlacq's original ten-figure table (where 7238085868 is printed instead of 7238085468), and is reproduced in Callet's (1795), Sherwin's, Hutton's, Babbage's, Taylor's, Shortrede's, and John Newton's tables; and Mr. Sang adds, that "it is remarkable that this error should have so long escaped detection, and all the more credit is due to its discoverer."

The discovery of the error, however, is not new, as the logarithm is printed correctly in Vega's 'Thesaurus,' &c., Lipsiæ, 1794 (folio), the error in Vlacq being pointed out in the preface. In the small seven-place editions of Vega and in Schrön's logarithms (1860) log 52943 is also printed correctly.

In tom. iv. of the *Annales de l'Observatoire Impérial de Paris* (1858), a table of errata in Vlacq's 'Arithmetica Logarithmica,' found by comparison with the great French MS. tables, is given by M. Lefort; it contains 452 errata (including those previously given in Vega's 'Thesaurus'), and the error in log 52943 is pointed out. I may mention that Lefort's errata list is intended to be supplementary to that in Vlacq's 'Arithmetica,' and at the last meeting of the Royal Astronomical Society I read a paper (printed in the *Monthly Notices* for May 1872), on the errors in Vlacq's table, in which I examined how far these lists taken together included all the known

errors; and chiefly by means of a copy of Vlacq in the library of the Royal Observatory at Greenwich, I was enabled to add seventeen errata, not given by Vlacq or Lefort. The total number of errors found in Vlacq thus amounts to 603, which probably includes very nearly all that exist; this cannot be regarded as a great number when it is considered that the table was the result of an original calculation, and that more than 2,100,000 printed figures are liable to error.

The occurrence of the error in log 52943 in Babbage's tables is very remarkable, as it is stated in the preface that they were read three times with the folio edition (1794) of Vega.

As I do not infer from Mr. Sang's letter that Dr. Whitcom claimed to have discovered the error in question himself, I have little doubt it came to light by a comparison between Sang's tables and Vega's errata list. In a seven-figure table the error only produces an additional error of 064 (the error, if 5 be taken as last figure, being + 468, and if 6, − 532), so that no one could detect it by the use of seven-figure logarithms. It may be remarked that although copies of Vlacq's tables are, comparatively speaking, common in England, as most of our chief libraries were in existence in 1628, in America they are very scarce; Vega's logarithms being almost exclusively used for work requiring ten figures.

There is another last-place error in Mr. Sang's tables, viz., in log 38962, the last figure of which should be 2 instead of 3; in this case, also, the logarithm in Vlacq is misprinted, being 5906413420 instead of 5906412420. This error is also pointed out by Vega.

In a paper, 'Notices respecting some errors common to many Tables of Logarithms,' (*Memoirs of the Astronomical Society*, tom. iii., 1829,) Babbage has pointed out six errors, of which this is one, that occur in most tables of logarithms; out of twenty-two the only tables free from them are the editions of Vega, the later editions of Callet and Hutton, and Babbage's own; none of the errors occur in Schrön's tables, but five out of the six are uncorrected in Shortrede's: all are noticed in Lefort's errata list.

It should be stated that of the errors found by Vega, some are corrected in the text, while some are given on a page following the introduction; the error in log 52943 is of the former class, that in log 38962 of the latter.

J. W. L. Glaisher.

University College School,
11 *June* 1872.

Mr. Sang is, I think, in error, when he states (*Athenæum*, June 8, 1872) that the mistake in Vlacq's tables has hitherto escaped detection. I had recently in my hands the late Mr. Babbage's copies of Hutton's Mathematical Tables (1804), and his own Tables (1829), in which are ink corrections of some logarithms which have been incorrectly given; and when I had seen Mr. Sang's letter, I again consulted these works, to see if the mistake in question had been noted by Mr. Babbage. The error is not indicated in the one or the other; but having access, through the courtesy of Col. Babbage,*

* I have this gentleman's permission to make use of his father's note.

to a copy of the Russian edition of Babbage's logarithms, I there came upon the desired correction, with a reference to the authority for the same: it is simply, "52943 is given as 7238086, it is 7238085, Bremiker's logs., pref.;" then the date of his noting it, 1862.

In Bremiker's edition (Berolini, 1852), which is only a "six-place" table, the log is given 723809, which is correct to that number of places; but in his 'Logarithmic Tables of Numbers and Trigonometrical Functions, by Baron von Vega, translated from the fortieth of Dr. Bremiker's thoroughly revised and enlarged edition, by W. L. F. Fischer, M.A., Prof. in the Univ. of St. Andrews (Berlin, 1857),' it is correctly given, 7238085; and this is the value also given in J. Dupuis' 'Tables de Logarithmes à sept décimales, d'après Callet, Vega, Bremiker, &c., deuxième tirage, Paris, 1863.' This puts back Mr. Sang's date about fifteen years. In this gentleman's seven-place logarithms (1871), we have 7238086, without the mark given by Mr. Babbage to indicate that the log was, as he thought, nearer 6 than 5 in the last place, showing that Mr. Sang had accepted Vlacq's results; and this we might have expected, as he informs us that his own independent calculations are confined to numbers ranging from 100000 to 200000. That his own work was carried on independently of Vlacq's tables, will appear from the following simple test: employing Vlacq's erroneous log to find log 105886; we should have it equal log 52943 + log 2, *i.e.* equal 4·7238085868 + ·3010299913, *i.e.* equal 5·0248385781, whereas it ought to be 5·0248385381. Now, had Mr. Sang merely copied Vlacq here, he would have given to seven-places 5·0248386 instead of 5·0248385, which he correctly gives.

R. TUCKER.

The following is Mr. Sang's reply.

Allow me to offer my thanks to your Correspondents for the interest they have taken in this matter, and particularly to Mr. Glaisher, for the indication of the error in log 38962, which had been caused by an omission to correct that logarithm in my copy of Vlacq, although noticed in my list. Mr. Glaisher inclines to the idea that Dr. Whitcom had found the mistake in some published list of errata. I rather think that he had found it in the course of his actuarial work. The logarithm in question is the middle one of the three 8003, 8086, 8167; the difference of the extremes being even, while the middle one is not half their sum. This can occur only once in seven-place logarithmic tables from 10000 to 100000, the solitary instance being in logs 12735, 6, 7, which are 9989; 0331; 0671; or, to 12 places, 89 49300; 30 50394; 71 48810. It does not again occur up to 200000. Now a careful computer, in the habit of looking to the difference both before and after, having to deal with some number between the above limits, could hardly fail to observe the anomaly.

As to the previous detection of the error, I could only consult the works on my own shelves, which include a copy of John Newton's eight-place table, kindly lent me by Mr. Peter Gray, and also one of Vega (1783), containing both of the errors.

I shall be grateful to any one having access to printed or MS. lists

of errors who may take the trouble to compare the same with my book, and shall gladly send him a copy for the first notice of each error. The second part of my work is quite independent of and has not yet been compared with any other table.

EDWARD SANG.

In a letter to Mr. Peter Gray, dated 7 Oct. 1872, Mr. Glaisher says, "I have no doubt Professor Whitcom did not claim to have found out the error in log 52943 himself, but obtained it (as I did that in log 38962) by comparison with Vega." He further mentions that Bruhns gives log 52943 correct, and that Hülsse's edition of Vega, 1840, has log 52943 *wrong*, altho' the previous editions have it right: also, that Dr. Bremiker in the preface to his 41st edition traces Babbage's error in log 52943 to Gardiner.

[In connection with this subject we may mention that our next Number will contain an article by Mr. Glaisher on "Errors in Tables of Logarithms of Numbers," being an abstract of a paper lately read by him before the Royal Astronomical Society.—ED. *J. I. A.*]

AMERICAN TEN-YEAR NONFORFEITURE POLICIES.

To the Editor of the Assurance Magazine.

SIR,—I send for publication, as likely to interest your readers, a statement of certain serious objections to the anomalous contract known in America as the Ten-Year Nonforfeiture Life Policy. These objections, always held and consistently acted upon by a few, have at last come to be widely recognized. The alteration of opinion has been manifested practically within a year or two past by changes in the plans of business of three, at least, of the largest companies.

A life policy calling for the payment of a limited number of annual premiums, is, of course, no novelty. Every writer on life contingencies furnishes, explicitly or implicitly, formulæ proper to be used in such cases. Secured contracts of this sort, forming an intermediate class between paid-up policies and those at annual premiums, are, in themselves, unobjectionable; presenting, apparently, no disadvantages which are not shared by one or other of those more usual forms of policy. By "secured contracts," I mean those which provide for forfeiture in case of lapse, or for such other penalty as will indemnify the society for whatever loss it may sustain by the default. Of this class are the ten-premium policies (now) written by some of the American companies, which promise the issue, in case of lapse, of a paid-up policy for "an equitable sum," to be determined by the company. This equitable sum, it must be presumed, is to be ascertained by making the present value of all liabilities on the new policy equal to that of those on the old, less proper damages for the breaking of the contract. Policies of this class, under which the society is, except in case of death, certain to receive all the premiums contemplated, or else a proper amount of profit-money out of the reserve value, may, for convenience, be called *secured ten-premium policies*.

A second variety of ten-premium policy was brought out some years ago by a prominent American company. At age 30, for

example, one could pay $45·83 for a paid-up policy of $100, with the privilege of taking another similar policy at the same price the following year, if desired, and so on, not exceeding ten years in all. (In practice, I believe, the first year's policy was so phrased as to cover the whole term, but the effect was as described—insuring $100 the first year, $200 the second, and so on.) This scheme seems to have come to grief speedily. Mr. Wright, in his report for 1864, remarks of it that it had, up to date, "tempted only one person."

A third and more attractive plan may be described as follows:—The first year's payment secures a paid-up policy for $100, *plus* a temporary assurance of $900 for the year. The next year, if the policyholder is inclined to go on, which he is permitted to do without re-examination, another payment secures $100 more, paid-up, and $800 temporary; the third year, if desired, an additional payment secures $100 more, paid-up, and $700 temporary; and so on for ten years. For distinction, I will call this the *ten-policy plan.* It seems to be embodied, with variations, in the prospectus of a new English company.

Ignoring for the present the injurious effects of the successive options allowed, let us see what premiums should be required for the benefits just described, beginning, for example, at age 50. The net premiums (Actuaries' or Old Experience, 4 per cent.) would be as follows:—

Year.	Age.	Premium on 100 dollars paid-up.	Premium on Temporary Assurance.	Total Net Premium.
1	50	48·19	13·79	61·98
2	51	49·31	13·00	62·31
3	52	50·45	12·08	62·53
4	53	51·60	11·02	62·62
5	54	52·76	9·77	62·53
6	55	53·93	8·33	62·26
7	56	55·12	6·67	61·79
8	57	56·31	4·75	61·06
9	58	57·51	2·54	60·05
10	59	58·73	0·00	58·73

These premiums are irregular, and cannot be reduced to uniformity; for equalization of premiums is not allowable when it puts the company out of pocket. If we attempt to find the uniform annual premium equivalent to this series, by dividing the sum of the present values of the natural premiums by the temporary annuity-due, we get $61·74, which is less by $0·24 than the sum required for the first year's risk alone. The net premiums required by the ten-policy plan are necessarily irregular, and this fact must be borne in mind by any one desiring to carry out that plan on scientific principles. Reserves, surrender values, and bonuses, must all be computed with express reference to this stubborn fact. Moreover, the commission to agents must be determined with due regard to the uncertainty attending the payment of the second and subsequent premiums. The agent must,

indeed, be paid separately for each of the ten premiums, as being, in fact, "single premiums" on ten different policies.

All these matters, however important, are still subordinate to the main question—Are such options safe? In theory they certainly are not. The opportunity afforded, for a term of years, to choose a larger or smaller amount of insurance, without re-examination, and without forfeiture or penalty of any sort in case the smaller amount is chosen, is so manifestly detrimental to the interests of the company granting it, that, before your readers at least, the subject need not be argued. The conclusions of a sound theory are not to be overthrown by any amount of apparently countervailing experience, so long as the testimony is merely negative, and no crucial case has arisen. Many thousands of policies too closely resembling those in question are now in force in America—above ten thousand in my own company alone—and no particular damage is traceable to the effects of the gratuitous options afforded under them. It would seem that such options do no great harm, on the whole, so long as the company granting them continues prosperous, and bonuses abound. *Poverty* probably causes as many bad lives, in proportion, to lapse as good. *Distrust* would operate chiefly among the best lives. A company doing a large business on what I have called the ten-policy plan, and becoming distressed while most of its policyholders had still several premiums to pay, would speedily find none but the very ignorant or the very sick continuing to avail themselves of the liberal options afforded to all.

The kind of policy to which I have just referred as being widely popular in America is that known as the *ten-year nonforfeiture policy.* It secures insurance for life, purchased by ten equal annual premiums, and provides that, in case of default in the payment of any premium (some companies make it any premium after the second), the policy shall stand as equivalent to, or may be exchanged for, a participating paid-up policy for a sum equal to as many tenth parts of the sum originally assured as there shall have been annual premiums paid. Legally, and as far as the policyholder knows, the contract corresponds throughout with the ten-policy plan just discussed. It is liable to the same objections, and to a good many more, owing to the careless way in which the business has been paid for, and also to erroneous computations of net premiums, reserves, and bonuses. Every one of the requirements I have mentioned, as attending the carrying out of the ten-policy plan on correct principles, has been utterly neglected. Agents have been paid large initial commissions for securing ten premiums, nine of them "in the bush." All the actuarial tables ever published in regard to these policies have been computed on the untenable assumption of uniform net premiums. In most cases, consequently, when a policyholder ceases to pay premiums, the reserve required to be held on the paid-up policy remaining, is larger than that actually in hand from past premiums. In this way he actually gets a special bonus for discontinuing his payments, instead of suffering a just penalty. Almost every company in America (and this is more especially the case with those dividing surplus on the Contribution Plan) has suffered innumerable practical inconveniences from this singular confusion of ideas and customs.

The ten-year nonforfeiture policy has always, for these reasons, been deemed objectionable in theory, by those who cared for theory. Its supposed acceptability preserved it from change for several years. Of late, however, it has declined in popularity; and certain of the larger companies have, as I have before intimated, made such alterations in their plans of business as to bring them into complete conformity with the only tenable theory—that of the secured ten-premium policy.

Not the least of the evils for which the ten-year nonforfeiture plan is responsible, is the encouragement it has given to the spread of loose views in regard to the lapsing of policies. There can be no doubt that the wholesale recommendation of this plan by the American companies, is the cause of much of the demoralization latterly prevailing in this regard. If the main object looked to in taking a policy is immunity in case of lapse, the holder cannot be expected to pay his premiums very persistently.

To sum up:

The *ten-year nonforfeiture plan*, as heretofore carried out in America, is a most vexatious arrangement, combining in practice two contradictory theories—the ten-policy theory, under which premiums may or may not be paid, and the secured ten-premium theory, under which all the premiums must be paid or something be forfeited.

The *secured ten-premium policy*, whether reserving the right to the company to fix the surrender-value according to circumstances, or itself specifying and guaranteeing proper surrender values, is a good contract for all parties concerned.

The *ten-policy plan* has never been rigorously carried out, at any rate not in America. It would have the advantage, common to all plans calling for large premiums, of securing at the start a class of risks of high average vitality. The only necessary difficulty attending it would be that, in case of adversity, the good lives would cease to pay premiums, while the deteriorated lives would keep their policies in full force. This difficulty, however, is insuperable.

I am, Sir,
Your obedient servant,

Northwestern Mutual Life Office. EMORY McCLINTOCK.
Milwaukee, U.S.A., 18 *Dec.* 1872.

*** In a letter which accompanied the above, Mr. McClintock makes some practical observations, from which we extract the following:—"Our present rule for ten-year life policies, is to issue in exchange, in case of lapse, a non-participating paid-up policy for an equitable sum, not less than two, three, &c., tenths. We regard the loss of participation as a penalty sufficient to make the new policies reasonably safe." "We have invested, this year alone, two millions of dollars at ten per cent. (semi-annually), on mortgages on property worth, in each case, about three times the amount of the loan, and have never lost a dollar of principal or interest."

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

On the Integral of Gompertz's Function for expressing the Values of Sums depending upon the contingency of life. By W. M. MAKEHAM, *Fellow of the Institute of Actuaries.*

GOMPERTZ'S Function for expressing the probability of a person surviving any given period, x, is $\frac{1}{g} \cdot g^{q^x}$. For the purpose of adapting this expression to the entire period of adult life I have added the factor ϵ^{-ax}. The two expressions become identical in *form* when the factor $\epsilon^{-\delta x}$ representing the effect of *discount* is introduced, and both may therefore be correctly designated by the name of "Gompertz's Function."

The interest excited by the publication of Gompertz's celebrated hypothesis, and the attention which has in consequence been bestowed upon it, have already resulted in the discovery of some curious and important properties of the function in question; and, as Mr. Woolhouse has pointed out, it is not improbable that the disclosure of others may reward the industry of those who choose to cultivate the field. The principal of those already discovered—viz., the property of uniform seniority—we owe to De Morgan. By the aid of this important property the number of

tables required in the computation of Life Contingencies became (as I have shown on a former occasion) materially reduced. For each mortality table, and each rate of interest, a complete set, from a single life up to any required number of joint lives, could be comprised in a comparatively moderate space, while under the old system the task would be attended with insuperable difficulties.

A further step in the same direction will be found in a paper by Mr. Woolhouse (see *Journal of the Institute of Actuaries* for July 1870), who suggested a contrivance by which the number of tables for any given rate of mortality could be reduced to a single series *for successive rates of interest*,—the annuity values for any given combination of ages being found by a process of *double* interpolation. It will be seen also, from the Editor's note at the end of the paper in question, that a similar idea had occurred to Mr. Meech, an eminent American actuary, and was communicated to me by that gentleman in December 1868.

By the last mentioned improvement we should be able to deduce the value of an annuity on any given combination of lives and at any given rate of interest, by means of a single table of double entry calculated for specific values of the constants g and q of the formula. But the table in question would be of no use for deducing the values of annuities according to a mortality table derived from other values of these constants. The table, however, which I now submit, and which, like that last referred to, is a single table of double entry, has this further important advantage, viz., that it is applicable to *all* values of the constants g and q whatever, and is therefore available for determining the values of annuities, not only for any given combination of lives and for any given rate of interest, *but also for any given rate of mortality.* That is to say, the advantage obtained by the law of uniform seniority (as modified according to the idea of Messrs. Meech and Woolhouse) in reference to a *single given mortality table,* is, by means of a contrivance which I have now to explain, extended so as to comprise all mortality tables whatever constructed according to Gompertz's Function.

The value of a continuous annuity on any given life or any combination of joint lives expressed in terms of Gompertz's Function, is

$$\frac{1}{g^{q^x}\epsilon^{-(a+\delta)x}}\int_x^\infty g^{q^x}.\epsilon^{-(a+\delta)x}dx.$$

Now it will be seen that in order to tabulate this function

completely we should require a table of no less than *four* variables—corresponding to the three constants g, q, and $(a+\delta)$, together with x, the inferior limit of integration. *Three* variables, altho' a somewhat troublesome matter, are at least *practicable* (as I have shown in a paper on this subject in vol. 16 of the *Journal*) by reason of the fact that space has *three* dimensions; but *four* involve a difficulty which would puzzle the genius of Euclid himself to surmount.

Among the contrivances available for facilitating the tabulation of definite integrals none are of more important use than that of changing the independent variable. In the case in hand, I put

$$z = x\log q + \log^2\tfrac{1}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$n = \frac{a+\delta}{\log q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the logarithms being common logarithms.

From (1) we get

$$dz = \log q \,.\, dx,$$

and substituting these values in the given function, the integral becomes

$$\frac{1}{\log q} \cdot \frac{1}{10^{-10^z} \,.\, \epsilon^{-nz}} \int_z^\infty 10^{-10^z} \,.\, \epsilon^{-nz} \,.\, dz,$$

in which it will be seen that by a very simple transformation we have succeeded in eliminating *two* of the arbitrary constants, and thus reducing the process of tabulation to one of *double entry* only.

The following is a proof of the transformation just effected.

$$g^{q^x} = 10^{\log g \,.\, 10^{\log q \,.\, x}}$$

$$= 10^{\log g \,.\, 10^{z - \log^2 \frac{1}{g}}}, \text{ from (1).}$$

But $\quad 10^{-\log^2 \frac{1}{g}} = 10^{-\log \,.\, \log \frac{1}{g}} = \dfrac{1}{10^{\log \,.\, \log \frac{1}{g}}} = \dfrac{1}{\log \frac{1}{g}} = -\dfrac{1}{\log g}$

$$\therefore \qquad g^{q^x} = 10^{\log g \,.\, 10^z \times -\frac{1}{\log g}}$$

$$= 10^{-10^z}.$$

Also $\epsilon^{-(a+\delta)x} = \epsilon^{-n\log q\left(\frac{z - \log_2 \frac{1}{g}}{\log q}\right)}$ from (1) and (2)

$$= \epsilon^{-nz} \epsilon^{n\log^2 \frac{1}{g}}$$

$$= \epsilon^{-nz} c \text{ [putting } \epsilon^{n\log^2 \frac{1}{g}} = c]$$

Hence we have

$$g^{q^x}\epsilon^{-(a+\delta)x}=10^{-10^z}.\epsilon^{-nz}.c$$

and as $dx=\dfrac{dz}{\log q}$, the integral becomes

$$\frac{1}{\log q}\cdot\frac{1}{10^{-10^z}.\epsilon^{-nz}.c}\int_z^\infty 10^{-10^z}.\epsilon^{-nz}.c.dz$$

from which we get the transformed integral above given, as the constant c from the way in which it is involved evidently disappears.

On a former occasion (see *Journal* vol. xiii, p. 349) I proposed the following transformation of the same integral, which would answer all the purposes of that just described, but which I think would be found more troublesome to calculate.

Put $$v=\left(\log_e\frac{1}{g}\right)\cdot q^x$$

and $$m=-\frac{a+d}{\log_e q}$$

whence $g^{q^x}(=\epsilon^{(\log_e g)q^x})=\epsilon^{-v}$. And observing that $-(a+\delta)=m\log_e q$, we have $\epsilon^{-(a+\delta)x}=\epsilon^{(\log_e q)mx}=q^{mx}=\dfrac{v^m}{(\log_e\frac{1}{g})^m}$. Again, $\dfrac{dv}{dx}=(\log_e\frac{1}{g}).q^x.\log_e q=v.\log_e q$. Hence, substituting, we have:

$$g^{q^x}\epsilon^{-(a+\delta)x}=\frac{\epsilon^{-v}.v^m}{(\log_e\frac{1}{g})^m}$$

and

$$\frac{1}{g^{q^x}\epsilon^{-(a+\delta)x}}\int_x^\infty g^{q^x}\epsilon^{-(a+\delta)x}dx=\frac{1}{\epsilon^{-v}.v^m.\log_e q}\int_v^\infty\epsilon^{-v}.v^{m-1}.dv$$

the second member of which equation (omitting the factors outside the symbol of integration) is the form of the well known gamma-function, or second Eulerian Integral.

Now it is a well known property of the last named integral that

$$\int\epsilon^{-v}.v^m.dv=-\epsilon^{-v}v^m+m\int\epsilon^{-v}v^{m-1}dv$$

whence the following equation is easily deduced:

$$\frac{v}{\epsilon^{-v}.v^{m+1}\log_e q}\int_v^\infty\epsilon^{-v}.v^m.dv=\frac{1}{\log_e q}+\frac{m}{\epsilon^{-v}.v^m.\log_e q}\int_v^\infty\epsilon^{-v}.v^{m-1}.dv$$

or

$$\frac{v}{\epsilon^{-v}.v^m.\log_e q}\int_v^\infty\epsilon^{-v}.v^{m-1}dv=\frac{1}{\log_e q}+\frac{m-1}{\epsilon^{-v}.v^{m-1}.\log_e q}\int_v^\infty\epsilon^{-v}.v^{m-2}dv$$

By means of these equations, having found the value of an annuity corresponding to any given value of m, we may with facility determine the annuity corresponding to $m\pm1$. It is by virtue of this property that we are able to limit the tabulated integral to values

of n from 1 to 2,—all other values being easily deduced from those found in the table.

Comparing the expression for the integral tabulated, viz.—

$$\log \cdot \frac{1}{10^{-10^z} . \epsilon^{-nz}} \int_z^\infty 10^{-10^z} . \epsilon^{-nz} . dz$$

with that representing the logarithmic values of annuities, viz.—

$$\log \frac{1}{g^{q^x} \epsilon^{-(a+\delta)x}} \int_x^\infty g^{q^x} \epsilon^{-(a+\delta)} . dx$$

it will be seen that the two are *identical in form.* This identity suggests a simple method of computing the former integral by the rules laid down by Mr. Woolhouse for the calculation of continuous annuities, which is the course I have found most convenient in the construction of the accompanying table.

The figures in antique type in the table are the divided differential coefficients of the function; and the process of interpolation is performed in accordance with the rules given in my paper on this subject in vol. xvi of the *Journal* (see page 111). The following examples will be sufficient to illustrate the process:—

Example I. Required the value of a continuous annuity at 5 per-cent on a life aged 30; according to the values of the constants deduced by Mr. Woolhouse from the new H^{MF} experience (see *Journal,* vol. xv, p. 405.)

$$\begin{array}{rll} \text{Here } \log \frac{1}{g} = & \cdot 0004121 & \qquad \log^2 \frac{1}{g} = \bar{4}\cdot 6150 \\ \log q = & \cdot 04 & \qquad \log q^{30} = 1\cdot 2 \\ a = \log_e a = & \cdot 00659 & \qquad z = \bar{3}\cdot 8150 \\ \delta = & \cdot 04879 & \\ a + \delta = & \cdot 05538 & \\ \frac{a+\delta}{\log q} = n = & 1\cdot 3845 & \end{array}$$

Nearest tabular value (corresponding to $z = 3\cdot 8$ and $n = 1\cdot 3$)

$$\begin{array}{lrrr} & & & = \bar{1}\cdot 81260 \\ - & 895 \times \cdot 150 = & -\ 134\cdot 2 & \\ - & 53 \times (\cdot 150)^2 = & -\ 1\cdot 2 & \\ - & 2360 \times \cdot 845 = & -1994\cdot 2 & -2129\cdot 6 = \bar{1}7870 \\ + & 44 \times (\cdot 845)^2 = & +\ 31\cdot 5 & \\ + & 71 \times \cdot 150 \times \cdot 845 = & +\ 9\cdot 0 & +40\cdot 5 = 40 \\ & & & = \bar{1}\cdot 79170 \\ & & \text{Deduct} & \log(\log q) = \bar{2}\cdot 60206 \\ & & & \text{log of annuity required} = \bar{1}\cdot 18964 \end{array}$$

When the required annuity is payable during the joint continuance of s lives, the value expressed in terms of Gompertz's function is

$$\frac{1}{g'^{q^x}\epsilon^{-(sa+\delta)x}}\int_x^\infty g'^{q^x}\epsilon^{-(sa+\delta)x}dx$$

where x is the youngest age and (d, d', d'' denoting the differences between x and the other ages respectively) g' is determined by the equation

$$\log\frac{1}{g'} = \log\frac{1}{g} \times (1+q^d+q^{d'}+\ldots\ldots)$$

or

$$\log^2\frac{1}{g'} = \log^2\frac{1}{g} + \log(1+q^d+q^{d'}+\ldots\ldots).$$

Example II. Required the value of a continuous annuity on two joint lives (aged 27 and 37 respectively) the rate of interest being 4 per-cent, and the constants of mortality being

$$\log^2\tfrac{1}{g} = \bar{4}\cdot407492$$
$$a = \cdot0081972$$
$$\log q = \cdot041667$$

Here $\log^2\frac{1}{g'} = \log^2\frac{1}{g} + \log(1+q^{10}) = \bar{4}\cdot4075 + \cdot5575 \qquad = \bar{4}\cdot9650$

$x \log q = \cdot041667 \times 27 = 1\cdot1250$

$\bar{2}\cdot0900 = z$

and $n = \dfrac{2a+\delta}{\log q} = \dfrac{\cdot01639 + \cdot03922}{\cdot041667} \qquad = 1\cdot3346 = n$

Nearest tabular value ($z = \bar{2}\cdot0$, $n = 1\cdot3$) $\qquad = \bar{1}\cdot79245$

$-1128 \times \cdot9$	$= -1015\cdot2$		
$-\ \ 65 \times (\cdot9)^2$	$= -\ \ 52\cdot7$		
$-2211 \times \cdot346$	$= -\ \ 765\cdot0$	$-1832\cdot9$	
$+\ \ 38 \times (\cdot346)^2$	$= +\ \ 4\cdot5$		
$+\ \ 77 \times \cdot9 \times \cdot346$	$= +\ \ 24\cdot0$	$+28\cdot5$	$= -1804$
			$\bar{1}\cdot77441$
		Deduct $\log(\log q)$	$= \bar{2}\cdot61979$
		log of annuity required	$= 1\cdot15462$

The corresponding annuity is 14·276. The value of the same annuity deduced independently from a private mortality table (constructed from the constants above given) is 14·275.

It will be observed that the trouble involved in the process of determining the required values arises solely from the limited

extent of the table, necessitating the use of two orders of differences, and may be entirely obviated by the construction of a table on a sufficiently extended scale.

Generally speaking, the values of annuities on a single life from 3 to 8 per-cent, and on as many as six joint lives at any rate up to 4 per-cent may be deduced directly from the table. This will no doubt be found sufficient for most practical purposes; but by means of the property of the gamma-function before adverted to the application of the table, if necessary, may be extended to cases beyond the above-mentioned limits.

I believe myself entitled to claim the credit of having first pointed out the equivalence of annuities payable at various intervals, when the period of the next payment is undetermined; and the consequent error of the practice of substituting the equivalent annual payment for Half-yearly and Quarterly premiums in the Valuations of Assurance Companies by the customary method of classification according to age. Mr. Woolhouse has since shown that in valuing (in the gross) a large number of independent annuities, payable at different intervals and at different periods of time, the employment of the continuous-annuity value is the strictly accurate mode of procedure (see his paper "On an Improved Theory of Annuities and Assurances" in vol. xv). Hence it would seem to follow that the tabulation of the continuous-annuity values, in lieu of the arbitrary "annually-payable" annuities, would be more consistent both with scientific accuracy and practical convenience. When the very common question is put, how many years' purchase is a given annuity worth?—without specifying either the *interval* or the *period* of payment,—the true answer, and the *only* true answer, is, the continuous-annuity value for the given age.

In the continuous system the "force of discount" becomes the *nominal* rate of interest; and would therefore be quoted in integers or the aliquot parts of integers. Thus (to take the case in Example II) in lieu of ·03922 . . . (*ad inf.*) which is the *force of discount* corresponding to 4 per-cent, we should take $\delta = \cdot 04$. The value of the reversion on a life aged x at the *nominal* rate of 4 per-cent (continuous) would therefore be $1 - \cdot 04 \times \bar{a}_x$, the numerical value of which expression may readily be ascertained by inspection from the table of annuity values. Many other advantages of a practical character would follow from the adoption of the continuous-annuity basis (as suggested by me in a letter published in the *Journal* for July 1861) for the valuation of life contingencies.

z	$n=1{\cdot}0$				$n=1{\cdot}1$				$n=1{\cdot}2$				$n=1{\cdot}3$				$n=1{\cdot}4$			
$\bar{4}$·0	$\bar{1}$·98182	−3795	+127	−4	$\bar{1}$·94510	−3552	+116	−4	$\bar{1}$·91070	−3331	+106	−3	$\bar{1}$·87841	−3129	+96	−3	$\bar{1}$·84805	−2946	+88	−3
	−184	+39	−5	0	−149	+30	−4	0	−121	+25	−3	0	−98	+20	−2	0	−81	+16	−2	0
	−9	+2	0	0	−8	+2	0	0	−7	+1	0	0	−6	+1	0	0	−5	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}$·1	$\bar{1}$·97989	−3754	+122	−4	$\bar{1}$·94353	−3520	+112	−3	$\bar{1}$·90942	−3305	+103	−3	$\bar{1}$·87737	−3108	+94	−3	$\bar{1}$·84719	−2929	+86	−3
	−204	+41	−4	0	−167	+33	−3	0	−136	+28	−3	0	−112	+22	−2	0	−94	+17	−2	0
	−10	+1	0	0	−9	+2	0	0	−8	+1	0	0	−7	+1	0	0	−6	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}$·2	$\bar{1}$·97775	−3712	+118	−4	$\bar{1}$·94177	−3485	+109	−3	$\bar{1}$·90798	−3276	+100	−3	$\bar{1}$·87618	−3085	+92	−3	$\bar{1}$·84622	−2911	+84	−2
	−225	+43	−4	0	−185	+35	−4	0	−153	+29	−3	0	−126	+24	−3	0	−105	+20	−2	0
	−11	+1	0	0	−10	+2	0	0	−9	+1	0	0	−8	+1	0	0	−7	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}$·3	$\bar{1}$·97539	−3668	+114	−4	$\bar{1}$·93982	−3448	+105	−3	$\bar{1}$·90636	−3246	+97	−3	$\bar{1}$·87484	−3060	+89	−3	$\bar{1}$·84510	−2890	+82	−2
	−249	+45	−3	0	−207	+38	−3	0	−73	+32	−3	0	−144	+26	−2	0	−120	+22	−3	0
	−13	+1	0	0	−12	+1	0	0	−10	+1	0	0	−9	+1	0	0	−8	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}$·4	$\bar{1}$·97277	−3622	+111	−3	$\bar{1}$·93763	−3409	+102	−3	$\bar{1}$·90453	−3213	+94	−3	$\bar{1}$·87331	−3033	+87	−2	$\bar{1}$·84382	−2867	+79	−2
	−276	+48	−4	0	−231	+40	−3	0	−194	+34	−3	0	−163	+28	−2	0	−137	+23	−2	0
	−14	+2	0	0	−13	+2	0	0	−11	+1	0	0	−10	+1	0	0	−9	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}$·5	$\bar{1}$·96987	−3572	+107	−3	$\bar{1}$·93519	−3367	+99	−3	$\bar{1}$·90248	−3178	+91	−3	$\bar{1}$·87158	−3004	+84	−2	$\bar{1}$·84236	−2843	+77	−2

x	$n=1\cdot5$				$n=1\cdot6$				$n=1\cdot7$				$n=1\cdot8$				$n=1\cdot9$			
$\bar{4}\cdot0$	$\bar{1}\cdot81943$	−2780	+80	−2	$\bar{1}\cdot79241$	−2629	+73	−2	$\bar{1}\cdot76683$	−2489	+66	−2	$\bar{1}\cdot74258$	−2363	+60	−2	$\bar{1}\cdot71953$	−2250	+54	−1
	−65	+13	−2	0	−54	+11	−2	0	−45	+9	−1	0	−37	+7	−1	0	−31	+5	−1	0
	−5	+1	0	0	−4	+1	0	0	−3	0	0	0	−3	0	0	0	−3	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}\cdot1$	$\bar{1}\cdot81873$	−2766	+78	−2	$\bar{1}\cdot79183$	−2617	+71	−2	$\bar{1}\cdot76635$	−2480	+65	−2	$\bar{1}\cdot74218$	−2356	+59	−2	$\bar{1}\cdot71919$	−2244	+53	−1
	−75	+14	−1	0	−62	+11	−1	0	−52	+10	−1	0	−43	+8	−1	0	−36	+6	−1	0
	−6	+1	0	0	−5	+1	0	0	−4	0	0	0	−4	+1	0	0	−3	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}\cdot2$	$\bar{1}\cdot81792$	−2751	+77	−2	$\bar{1}\cdot79116$	−2605	+70	−2	$\bar{1}\cdot76579$	−2470	+64	−2	$\bar{1}\cdot74171$	−2347	+58	−2	$\bar{1}\cdot71880$	−2238	+52	−1
	−87	+17	−2	0	−72	+13	−1	0	−60	+10	−1	0	−51	+9	−1	0	−43	+7	−1	0
	−6	0	0	0	−6	+1	0	0	−5	+1	0	0	−4	0	0	0	−4	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}\cdot3$	$\bar{1}\cdot81699$	−2734	+75	−2	$\bar{1}\cdot79038$	−2591	+69	−2	$\bar{1}\cdot76514$	−2459	+63	−2	$\bar{1}\cdot74116$	−2338	+57	−2	$\bar{1}\cdot71833$	−2231	+51	−1
	−100	+18	−2	0	−84	+15	−2	0	−71	+12	−1	0	−61	+10	−1	0	−51	+8	−1	0
	−7	0	0	0	−7	+1	0	0	−6	+1	0	0	−5	0	0	0	−5	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}\cdot4$	$\bar{1}\cdot81592$	−2716	+73	−2	$\bar{1}\cdot78947$	−2575	+67	−2	$\bar{1}\cdot76437$	−2446	+62	−2	$\bar{1}\cdot74050$	−2328	+56	−2	$\bar{1}\cdot71777$	−2222	+50	−1
	−116	+20	−2	0	−98	+16	−1	0	−83	+13	−2	0	−71	+11	−1	0	−61	+9	−1	0
	−8	+1	0	0	−7	+1	0	0	−6	+1	0	0	−5	0	0	0	−5	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}\cdot5$	$\bar{1}\cdot81468$	−2695	+71	−2	$\bar{1}\cdot78842$	−2558	+66	−2	$\bar{1}\cdot76348$	−2432	+60	−2	$\bar{1}\cdot73974$	−2317	+55	−1	$\bar{1}\cdot71711$	−2212	+49	−1

s	$n=1\cdot0$				$n=1\cdot1$				$n=1\cdot2$				$n=1\cdot3$				$n=1\cdot4$			
$\bar{4}$·5	$\bar{1}$·96987	−3572	+107	−3	$\bar{1}$·93519	−3367	+99	−3	$\bar{1}$·90248	−3178	+91	−3	$\bar{1}$·87158	−3004	+84	−2	$\bar{1}$·84236	−2843	+77	−2
	−305	+51	−4	0	−258	+43	−4	0	−218	+36	−3	0	−185	+31	−3	0	−157	+26	−2	0
	−15	+1	0	0	−14	+1	0	0	−13	+2	0	0	−11	+1	0	0	−10	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}$·6	$\bar{1}$·96667	−3520	+103	−3	$\bar{1}$·93247	−3323	+95	−2	$\bar{1}$·90017	−3140	+88	−3	$\bar{1}$·86962	−2972	+81	−2	$\bar{1}$·84069	−2816	+75	−2
	−337	+54	−4	0	−286	+46	−4	0	−245	+39	−3	0	−209	+33	−3	0	−179	+28	−2	0
	−17	+2	0	0	−16	+2	0	0	−14	+1	0	0	−13	+1	0	0	−12	+1	0	0
	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}$·7	$\bar{1}$·96312	−3464	+99	−3	$\bar{1}$·92944	−3275	+91	−2	$\bar{1}$·89758	−3100	+84	−2	$\bar{1}$·86740	−2938	+78	−2	$\bar{1}$·83878	−2787	+73	−2
	−373	+58	−5	0	−319	+49	−4	0	−275	+42	−3	0	−236	+36	−3	0	−203	+30	−3	0
	−19	+2	0	0	−18	+1	0	0	−16	+2	0	0	−15	+2	0	0	−13	+2	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{4}$·8	$\bar{1}$·95919	−3404	+94	−3	$\bar{1}$·92606	−3225	+87	−2	$\bar{1}$·89466	−3056	+81	−2	$\bar{1}$·86488	−2900	+75	−2	$\bar{1}$·83661	−2755	+70	−2
	−414	+61	−5	0	−357	+52	−3	0	−309	+45	−3	0	−267	+39	−3	0	−232	+33	−3	0
	−21	+1	0	0	−20	+2	0	0	−18	+1	0	0	−17	+1	0	0	−15	+1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{4}$·9	$\bar{1}$·95483	−3342	+89	−2	$\bar{1}$·92228	−3171	+83	−2	$\bar{1}$·89138	−3010	+78	−2	$\bar{1}$·86203	−2860	+72	−2	$\bar{1}$·83413	−2721	+67	−2
	−457	+64	−4	0	−398	+55	−4	0	−347	+48	−4	0	−302	+41	−3	0	−265	+35	−2	0
	−23	+1	0	0	−22	+2	0	0	−20	+1	0	0	−19	+1	0	0	−17	+1	0	0
	1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{8}$·0	$\bar{1}$·95002	−3277	+85	−2	$\bar{1}$·91807	−3114	+79	−2	$\bar{1}$·88770	−2961	+74	−2	$\bar{1}$·85881	−2818	+69	−2	$\bar{1}$·83130	−2685	+65	−1

s	$n=1{\cdot}5$				$n=1{\cdot}6$				$n=1{\cdot}7$				$n=1{\cdot}8$				$n=1{\cdot}9$			
$\bar{4}{\cdot}5$	$\bar{1}{\cdot}81468$	−2695	+71	−2	$\bar{1}{\cdot}78842$	−2558	+66	−2	$\bar{1}{\cdot}76348$	−2432	+60	−2	$\bar{1}{\cdot}73974$	−2317	+55	−1	$\bar{1}{\cdot}71711$	−2212	+49	−1
	−134	+22	−2	0	−114	+19	−2	0	−97	+15	−1	0	−83	+12	−1	0	−72	+10	0	0
	−9	+1	0	0	−8	0	0	0	−8	+1	0	0	−7	+1	0	0	−6	+1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}{\cdot}6$	$\bar{1}{\cdot}81325$	−2672	+69	−2	$\bar{1}{\cdot}78720$	−2539	+64	−2	$\bar{1}{\cdot}76243$	−2416	+59	−2	$\bar{1}{\cdot}73884$	−2304	+54	−1	$\bar{1}{\cdot}71633$	−2201	+49	−1
	−152	+24	−2	0	−130	+19	−2	0	−113	+15	−1	0	−98	+14	0	0	−85	+13	0	0
	−10	0	0	0	−10	+1	0	0	−9	+1	0	0	−8	+1	0	0	−7	+1	0	0
	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{4}{\cdot}7$	$\bar{1}{\cdot}81162$	−2648	+67	−2	$\bar{1}{\cdot}78579$	−2519	+62	−1	$\bar{1}{\cdot}76121$	−2400	+58	−1	$\bar{1}{\cdot}73778$	−2289	+54	−1	$\bar{1}{\cdot}71541$	−2187	+49	−1
	−176	+26	−2	0	−151	+22	−2	0	−131	+19	−2	0	−114	+15	−2	0	−101	+14	0	0
	−12	+1	0	0	−11	+1	0	0	−10	+1	0	0	−9	+1	0	0	−8	+2	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{4}{\cdot}8$	$\bar{1}{\cdot}80973$	−2621	+65	−1	$\bar{1}{\cdot}78416$	−2496	+60	−1	$\bar{1}{\cdot}75979$	−2380	+56	−1	$\bar{1}{\cdot}73654$	−2273	+52	−1	$\bar{1}{\cdot}71432$	−2171	+49	−1
	−201	+28	−2	0	−175	+24	−2	0	−153	+21	−2	0	−134	+18	−1	0	−117	+15	−1	0
	−14	+1	0	0	−13	+1	0	0	−12	+1	0	0	−11	+1	0	0	−10	+1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{4}{\cdot}9$	$\bar{1}{\cdot}80757$	−2592	+63	−1	$\bar{1}{\cdot}78227$	−2471	+58	−1	$\bar{1}{\cdot}75813$	−2358	+54	−1	$\bar{1}{\cdot}73508$	−2254	+51	−1	$\bar{1}{\cdot}71304$	−2155	+48	−1
	−231	+31	−3	0	−203	+26	−2	0	−178	+23	−2	0	−157	+20	−2	0	−139	+16	−2	0
	−16	+1	0	0	−15	+2	0	0	−13	+1	0	0	−12	+1	0	0	−12	+1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}{\cdot}0$	$\bar{1}{\cdot}80509$	−2560	+60	−1	$\bar{1}{\cdot}78008$	−2443	+56	−1	$\bar{1}{\cdot}75621$	−2334	+52	−1	$\bar{1}{\cdot}73338$	−2233	+49	−1	$\bar{1}{\cdot}71153$	−2138	+46	−1

z	$n=1·0$				$n=1·1$				$n=1·2$				$n=1·3$				$n=1·4$			
$\bar{3}$·0	$\bar{1}$·95002	−3277	+85	−2	$\bar{1}$·91807	−3114	+79	−2	$\bar{1}$·88770	−2961	+74	−2	$\bar{1}$.85881	−2818	+69	−2	$\bar{1}$·83130	−2685	+65	−1
	−507	+67	−5	0	−444	+58	−4	0	−389	+51	−4	0	−342	+44	−3	0	−301	+38	−3	0
	−26	+1	0	0	−24	+2	0	0	−23	+1	0	0	−21	+1	0	0	−19	+2	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}$·1	$\bar{1}$·94468	−3209	+80	−2	$\bar{1}$·91338	−3054	+75	−2	$\bar{1}$.88357	−2909	+70	−1	$\bar{1}$·85517	−2773	+66	−1	$\bar{1}$·82809	−2645	+62	−1
	−560	+70	−4	0	−496	+61	−4	0	−438	+54	−3	0	−387	+47	−3	0	−343	+41	−3	0
	−29	+2	0	0	−27	+2	0	0	−25	+2	0	0	−24	+2	0	0	−22	+2	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}$·2	$\bar{1}$·93878	−3137	+76	−2	$\bar{1}$·90814	−2991	+71	−1	$\bar{1}$·87893	−2853	+67	−1	$\bar{1}$·85105	−2724	+63	−1	$\bar{1}$·82443	−2602	+59	−1
	−621	+73	−4	0	−552	+64	−3	0	−492	+57	−3	0	−437	+50	−3	0	−390	+44	−3	0
	−32	+1	0	0	−30	+2	0	0	−28	+2	0	0	−27	+2	0	0	−25	+2	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}$·3	$\bar{1}$·93224	−3063	+72	−2	$\bar{1}$·90231	−2925	+68	−1	$\bar{1}$·87372	−2794	+64	−1	$\bar{1}$·84640	−2672	+60	−1	$\bar{1}$·82027	−2556	+56	−1
	−687	+75	−4	0	−616	+67	−3	0	−551	+60	−3	0	−494	+54	−4	0	−444	+47	−3	0
	−35	+1	0	0	−33	+1	0	0	−32	+2	0	0	−30	+2	0	0	−28	+2	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}$·4	$\bar{1}$·92501	−2986	+68	−2	$\bar{1}$·89581	−2856	+64	−1	$\bar{1}$·86788	−2732	+60	−1	$\bar{1}$·84115	−2616	+56	−1	$\bar{1}$·81554	−2507	+53	−1
	−761	+79	−4	0	−685	+71	−3	0	−618	+64	−3	0	−557	+58	−4	0	−503	+51	−3	0
	−39	+2	0	0	−37	+2	0	0	−35	+1	0	0	−34	+2	0	0	−32	+2	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}$·5	$\bar{1}$·91700	−2905	+64	−1	$\bar{1}$·88858	−2783	+60	−1	$\bar{1}$·86134	−2667	+56	−1	$\bar{1}$·83523	−2557	+53	−1	$\bar{1}$·81018	−2454	+50	−1

z	$n=1\cdot5$				$n=1\cdot6$				$n=1\cdot7$				$n=1\cdot8$				$n=1\cdot9$			
$\bar{3}\cdot0$	$\bar{1}\cdot80509$	−2560	+60	−1	$\bar{1}\cdot78008$	−2443	+56	−1	$\bar{1}\cdot75621$	−2334	+52	−1	$\bar{1}\cdot73338$	−2233	+49	−1	$\bar{1}\cdot71153$	−2138	+46	−1
	−265	+34	−2	0	−233	+29	−2	0	−207	+25	−1	0	−184	+22	−2	0	−164	+19	−2	0
	−18	+1	0	0	−17	+1	0	0	−16	+1	0	0	−14	+1	0	0	−13	+1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}\cdot1$	$\bar{1}\cdot80225$	−2525	+58	−1	$\bar{1}\cdot77757$	−2413	+54	−1	$\bar{1}\cdot75397$	−2308	+51	−1	$\bar{1}\cdot73139$	−2210	+47	−1	$\bar{1}\cdot70975$	−2118	+44	−1
	−304	+36	−2	0	−271	+32	−2	0	−241	+28	−2	0	−215	+24	−1	0	−192	+21	−2	0
	−21	+1	0	0	−19	+1	0	0	−18	+1	0	0	−17	+2	0	0	−15	+1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}\cdot2$	$\bar{1}\cdot79899$	−2488	+56	−1	$\bar{1}\cdot77466$	−2380	+52	−1	$\bar{1}\cdot75137$	−2279	+49	−1	$\bar{1}\cdot72906$	−2184	+46	−1	$\bar{1}\cdot70767$	−2096	+42	−1
	−348	+39	−2	0	−311	+35	−2	0	−279	+31	−2	0	−250	+27	−2	0	−226	+23	−1	0
	−24	+2	0	0	−22	+1	0	0	−20	+1	0	0	−19	+1	0	0	−18	+1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}\cdot3$	$\bar{1}\cdot79526$	−2447	+54	−1	$\bar{1}\cdot77132$	−2344	+50	−1	$\bar{1}\cdot74837$	−2247	+47	−1	$\bar{1}\cdot72636$	−2156	+44	−1	$\bar{1}\cdot70522$	−2072	+41	−1
	−398	+42	−3	0	−358	+37	−2	0	−323	+33	−2	0	−292	+29	−2	0	−264	+26	−2	0
	−27	+2	0	0	−25	+1	0	0	−24	+1	0	0	−22	+1	0	0	−20	+1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}\cdot4$	$\bar{1}\cdot79100$	−2403	+51	−1	$\bar{1}\cdot76747$	−2305	+48	−1	$\bar{1}\cdot74489$	−2213	+45	0	$\bar{1}\cdot72321$	−2126	+42	0	$\bar{1}\cdot70237$	−2045	+39	0
	−455	+45	−3	0	−412	+41	−3	0	−374	+36	−2	0	−340	+33	−2	0	−309	+30	−1	0
	−30	+2	0	0	−28	+1	0	0	−27	+2	0	0	−25	+1	0	0	−24	+1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}\cdot5$	$\bar{1}\cdot78614$	−2356	+48	0	$\bar{1}\cdot76306$	−2263	+45	−1	$\bar{1}\cdot74087$	−2175	+43	0	$\bar{1}\cdot71955$	−2092	+40	0	$\bar{1}\cdot69903$	−2014	+38	0

z	$n=1{\cdot}0$				$n=1{\cdot}1$				$n=1{\cdot}2$				$n=1{\cdot}3$				$n=1{\cdot}4$			
$\bar{3}{\cdot}5$	$\bar{1}{\cdot}91700$	−2905	+64	−1	$\bar{1}{\cdot}88858$	−2783	+60	−1	$\bar{1}{\cdot}86134$	−2667	+56	−1	$\bar{1}{\cdot}83523$	−2557	+53	−1	$\bar{1}{\cdot}81018$	−2454	+50	−1
	−842	+83	−4	0	−763	+75	−4	0	−692	+68	−3	0	−628	+61	−3	0	−571	+54	−3	0
	−43	+1	0	0	−41	+2	0	0	−39	+2	0	0	−38	+1	0	0	−36	+2	0	0
	−1	0	0	0	−2	0	0	0	−2	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}{\cdot}6$	$\bar{1}{\cdot}90814$	−2821	+60	−1	$\bar{1}{\cdot}88052$	−2706	+56	−1	$\bar{1}{\cdot}85401$	−2597	+53	−1	$\bar{1}{\cdot}82856$	−2495	+50	−1	$\bar{1}{\cdot}80410$	−2398	+47	−1
	−931	+86	−4	0	−850	+78	−4	0	−775	+71	−4	0	−708	+64	−3	0	−647	+59	−2	0
	−47	+2	0	0	−45	+2	0	0	−44	+1	0	0	−42	+2	0	0	−40	+2	0	0
	−2	0	0	0	−1	0	0	0	−1	0	0	0	−2	0	0	0	−2	0	0	0
$\bar{3}{\cdot}7$	$\bar{1}{\cdot}89834$	−2733	+56	−1	$\bar{1}{\cdot}87156$	−2626	+52	−1	$\bar{1}{\cdot}84581$	−2525	+49	−1	$\bar{1}{\cdot}82104$	−2429	+47	−1	$\bar{1}{\cdot}79721$	−2337	+45	−1
	−1029	+89	−4	0	−945	+81	−3	0	−867	+74	−3	0	−796	+67	−3	0	−732	+62	−3	0
	−52	+2	0	0	−50	+2	0	0	−49	+2	0	0	−47	+2	0	0	−45	+1	0	0
	−2	0	0	0	−2	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{3}{\cdot}8$	$\bar{1}{\cdot}88751$	−2642	+52	−1	$\bar{1}{\cdot}86159$	−2543	+49	−1	$\bar{1}{\cdot}83664$	−2449	+46	−1	$\bar{1}{\cdot}81260$	−2360	+44	−1	$\bar{1}{\cdot}78943$	−2274	+42	−1
	−1139	+92	−4	0	−1050	+84	−4	0	−969	+77	−4	0	−895	+71	−3	0	−827	+65	−3	0
	−57	+2	0	0	−56	+2	0	0	−54	+2	0	0	−53	+2	0	0	−51	+2	0	0
	−2	0	0	0	−1	0	0	0	−2	0	0	0	−2	0	0	0	−2	0	0	0
$\bar{3}{\cdot}9$	$\bar{1}{\cdot}87553$	−2548	+48	−1	$\bar{1}{\cdot}85052$	−2456	+45	−1	$\bar{1}{\cdot}82639$	−2370	+42	−1	$\bar{1}{\cdot}80310$	−2287	+41	−1	$\bar{1}{\cdot}78063$	−2207	+39	−1
	−1258	+95	−5	0	−1167	+88	−4	0	−1083	+81	−3	0	−1005	+74	−3	0	−934	+68	−3	0
	−63	+1	0	0	−62	+1	0	0	−60	+2	0	0	−58	+2	0	0	−56	+2	0	0
	−2	0	0	0	−2	0	0	0	−2	0	0	0	−2	0	0	0	−2	0	0	0
$\bar{2}{\cdot}0$	$\bar{1}{\cdot}86230$	−2452	+43	−1	$\bar{1}.83821$	−2367	+41	−1	$\bar{1}{\cdot}81494$	−2287	+39	−1	$\bar{1}{\cdot}79245$	−2211	+38	−1	$\bar{1}.77071$	−2137	+36	−1

x	$n=1\cdot5$				$n=1\cdot6$				$n=1\cdot7$				$n=1\cdot8$				$n=1\cdot9$			
$\bar{3}\cdot5$	$\bar{1}\cdot78614$ −520 −34 −1	−2356 +49 +2 0	+48 −3 0 0	0 0 0 0	$\bar{1}\cdot76306$ −474 −32 −1	−2263 +45 +1 0	+45 −2 0 0	−1 0 0 0	$\bar{1}\cdot74087$ −431 −31 −1	−2175 +39 +2 0	+43 −2 0 0	0 0 0 0	$\bar{1}\cdot71955$ −395 −29 −1	−2092 +36 +1 0	+40 −1 0 0	0 0 0 0	$\bar{1}\cdot69903$ −361 −28 −1	−2014 +33 +1 0	+38 −1 0 0	0 0 0 0
$\bar{3}\cdot6$	$\bar{1}\cdot78059$ −591 −38 −2	−2305 +53 +1 0	+45 −3 0 0	0 0 0 0	$\bar{1}\cdot75799$ −541 −37 −2	−2217 +47 +2 0	+43 −3 0 0	−1 0 0 0	$\bar{1}\cdot73624$ −496 −35 −2	−2134 +42 +2 0	+41 −3 0 0	−1 0 0 0	$\bar{1}\cdot71530$ −456 −33 −2	−2055 +39 +1 0	+39 −2 0 0	−1 0 0 0	$\bar{1}\cdot69513$ −419 −32 −2	−1980 +36 +1 0	+37 −1 0 0	0 0 0 0
$\bar{3}\cdot7$	$\bar{1}\cdot77428$ −674 −43 −2	−2251 +57 +1 0	+42 −2 0 0	0 0 0 0	$\bar{1}\cdot75219$ −619 −42 −2	−2168 +50 +2 0	+40 −2 0 0	0 0 0 0	$\bar{1}\cdot73091$ −572 −40 −2	−2090 +46 +2 0	+38 −2 0 0	0 0 0 0	$\bar{1}\cdot71039$ −528 −38 −2	−2015 +43 +1 0	+37 −2 0 0	−1 0 0 0	$\bar{1}\cdot69060$ −487 −37 −2	−1943 +39 +1 0	+36 −2 0 0	0 0 0 0
$\bar{3}\cdot8$	$\bar{1}\cdot76709$ −765 −48 −2	−2193 +60 +1 0	+40 −3 0 0	0 0 0 0	$\bar{1}\cdot74556$ −708 −48 −2	−2116 +53 +3 0	+38 −2 0 0	−1 0 0 0	$\bar{1}\cdot72477$ −656 −45 −2	−2042 +50 +2 0	+36 −2 0 0	0 0 0 0	$\bar{1}\cdot70471$ −609 −43 −2	−1971 +46 +1 0	+35 −2 0 0	−1 0 0 0	$\bar{1}\cdot68534$ −565 −42 −2	−1903 +42 +1 0	+34 −2 0 0	0 0 0 0
$\bar{3}\cdot9$	$\bar{1}\cdot75894$ −868 −55 −2	−2132 +63 +1 0	+37 −3 0 0	−1 0 0 0	$\bar{1}\cdot73798$ −808 −53 −2	−2060 +58 +2 0	+36 −3 0 0	0 0 0 0	$\bar{1}\cdot71774$ −753 −51 −2	−1990 +53 +2 0	+34 −2 0 0	−1 0 0 0	$\bar{1}\cdot69817$ −702 −49 −2	−1924 +49 +1 0	+33 −2 0 0	−1 0 0 0	$\bar{1}\cdot67925$ −655 −48 −2	−1860 +45 +1 0	+32 −2 0 0	0 0 0 0
$\bar{2}\cdot0$	$\bar{1}\cdot74969$	−2068	+34	0	$\bar{1}\cdot72935$	−2000	+33	0	$\bar{1}\cdot70968$	−1935	+32	−1	$\bar{1}\cdot69064$	−1874	+31	0	$\bar{1}\cdot67220$	−1814	+30	0

x	$n=1·0$				$n=1·1$				$n=1·2$				$n=1·3$				$n=1·4$			
$\bar{2}$·0	$\bar{1}$·86230	−2452	+43	−1	$\bar{1}$·83821	−2367	+41	−1	$\bar{1}$·81494	−2287	+39	−1	$\bar{1}$·79245	−2211	+38	−1	$\bar{1}$·77071	−2137	+36	−1
	−1390	+98	−4	0	−1296	+91	−4	0	−1209	+84	−3		−1128	+77	−3	0	−1053	+71	−3	0
	−69	+2	0	0	−68	+1	0	0	−66	+1			−65	+2	0	0	−63	+1	0	0
	−2	0	0	0	−1	0	0	0	−2				−2	0	0	0	−2	0	0	0
$\bar{2}$·1	$\bar{1}$·84769	−2352	+39	0	$\bar{1}$·82456	−2276	+37	0	$\bar{1}$·80217	−2203	+36	0	$\bar{1}$·78050	−2132	+35	0	$\bar{1}$·75953	−2065	+33	0
	−1534	+101	−4	0	−1440	+94	−3	0	−1348	+87	−3	0	−1263	+81	−3	0	−1185	+75	−2	0
	−76	+1	0	0	−73	+1	0	0	−73	+2	0	0	−71	+1	0	0	−70	+2	0	0
	−2	0	0	0	−2	0	0	0	−2	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{2}$·2	$\bar{1}$·83157	−2250	+35	0	$\bar{1}$·80941	−2181	+34	0	$\bar{1}$·78794	−2114	+33	0	$\bar{1}$·76713	−2050	+32	0	$\bar{1}$·74695	−1988	+31	0
	−1692	+102	−3	0	−1593	+96	−3	0	−1500	+90	−3	0	−1413	+84	−3	0	−1331	+79	−3	0
	−83	+1	0	0	−81	+1	0	0	−80	+1	0	0	−79	+2	0	0	−78	+2	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{2}$·3	$\bar{1}$·81379	−2147	+32	0	$\bar{1}$·79264	−2084	+31	0	$\bar{1}$·77211	−2023	+30	0	$\bar{1}$·75218	−1964	+29	0	$\bar{1}$·73283	−1907	+28	0
	−1865	+105	−3	0	−1764	+99	−3	0	−1668	+92	−3	0	−1579	+86	−2	0	−1495	+81	−2	0
	−91	+1	0	0	−89	+1	0	0	−88	+1	0	0	−87	+1	0	0	−86	+1	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{2}$·4	$\bar{1}$·79420	−2041	+29	0	$\bar{1}$·77408	−1984	+28	0	$\bar{1}$·75452	−1930	+27	0	$\bar{1}$·73549	−1877	+27	0	$\bar{1}$·71699	−1825	+26	0
	−2053	+107	−3	0	−1950	+100	−3	0	−1853	+94	−2	0	−1761	+89	−3	0	−1674	+84	−3	0
	−99	+1	0	0	−98	+1	0	0	−97	+2	0	0	−95	+2	0	0	−94	+2	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{2}$·5	$\bar{1}$·77265	−1933	+26	0	$\bar{1}$·75357	−1883	+25	0	$\bar{1}$·73499	−1834	+25	0	$\bar{1}$·71690	−1786	+24	0	$\bar{1}$·69928	−1739	+23	0

z	*n* = 1·5				*n* = 1·6				*n* = 1·7				*n* = 1·8				*n* = 1·9			
$\bar{2}$·0	$\bar{1}$·74969 −985 −61 −2	−2068 +67 +2 0	+34 −2 0 0	0 0 0 0	$\bar{1}$·72935 −920 −59 −2	−2000 +62 +1 0	+33 −3 0 0	0 0 0 0	$\bar{1}$·70968 −861 −58 −2	−1935 +57 +1 0	+32 −3 0 0	−1 0 0 0	$\bar{1}$·69064 −807 −56 −2	−1874 +53 +2 0	+31 −2 0 0	0 0 0 0	$\bar{1}$·67220 −756 −54 −2	−1814 +49 +2 0	+30 −2 0 0	0 0 0 0
$\bar{2}$·1	$\bar{1}$·73921 −1113 −68 −2	−1999 +69 +2 0	+32 −3 0 0	0 0 0 0	$\bar{1}$·71954 −1047 −66 −2	−1937 +65 +2 0	+30 −2 0 0	0 0 0 0	$\bar{1}$·70047 −984 −65 −2	−1877 +60 +2 0	+29 −2 0 0	0 0 0 0	$\bar{1}$·68199 −925 −63 −2	−1819 +55 +2 0	+28 −2 0 0	0 0 0 0	$\bar{1}$·66408 −872 −61 −2	−1763 +51 +2 0	+28 −2 0 0	0 0 0 0
$\bar{2}$·2	$\bar{1}$·72738 −1256 −76 −3	−1928 +74 +2 0	+29 −2 0 0	0 0 0 0	$\bar{1}$·70839 −1185 −74 −3	−1870 +69 +1 0	+28 −3 0 0	0 0 0 0	$\bar{1}$·68997 −1119 −73 −3	−1815 +64 +2 0	+27 −2 0 0	0 0 0 0	$\bar{1}$·67209 −1057 −71 −3	−1762 +59 +2 0	+26 −2 0 0	0 0 0 0	$\bar{1}$·65473 −1000 −69 −3	−1711 +55 +2 0	+26 −2 0 0	0 0 0 0
$\bar{2}$·3	$\bar{1}$·71403 −1416 −84 −3	−1852 +76 +2 0	+27 −3 0 0	0 0 0 0	$\bar{1}$·69577 −1342 −82 −3	−1800 +71 +2 0	+25 −2 0 0	0 0 0 0	$\bar{1}$·67802 −1272 −81 −3	−1749 +67 +1 0	+25 −2 0 0	0 0 0 0	$\bar{1}$·66078 −1207 −79 −3	−1701 +64 +1 0	+24 −2 0 0	0 0 0 0	$\bar{1}$·64401 −1146 −78 −3	−1653 +60 +1 0	+24 −2 0 0	9 0 0 0
$\bar{2}$·4	$\bar{1}$·69900 −1592 −93 −3	−1774 +78 +2 0	+24 −2 0 0	0 0 0 0	$\bar{1}$·68150 −1516 −91 −3	−1727 +74 +2 0	+23 −2 0 0	0 0 0 0	$\bar{1}$·66446 −1443 −90 −3	−1681 +70 +2 0	+23 −2 0 0	0 0 0 0	$\bar{1}$·64788 −1375 −88 −3	−1636 +67 +1 0	+22 −2 0 0	0 0 0 0	$\bar{1}$·63174 −1310 −87 −3	−1592 +63 +1 0	+22 −2 0 0	0 0 0 0
$\bar{2}$·5	$\bar{1}$·68212	−1694	+22	0	$\bar{1}$·66540	−1651	+21	0	$\bar{1}$·64910	−1609	+21	0	$\bar{1}$·63322	−1568	+20	0	$\bar{1}$·61774	−1528	+20	0

z	$n=1\cdot0$				$n=1\cdot1$				$n=1\cdot2$				$n=1\cdot3$				$n=1\cdot4$			
$\bar{2}\cdot5$	$\bar{1}\cdot77265$	-1933	+26	0	$\bar{1}\cdot75357$	-1883	+25	0	$\bar{1}\cdot73499$	-1834	+25	0	$\bar{1}\cdot71690$	-1786	+24	0	$\bar{1}\cdot69928$	-1739	+23	0
	-2260	+109	-3	0	-2155	+103	-3	0	-2055	+97	-3	0	-1961	+92	-3	0	-1872	+87	-2	0
	-107	+1	0	0	-106	+1	0	0	-105	+1	0	0	-104	+1	0	0	-103	+1	0	0
	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0
$\bar{2}\cdot6$	$\bar{1}\cdot74895$	-1823	+23	0	$\bar{1}\cdot73093$	-1779	+22	0	$\bar{1}\cdot71336$	-1736	+22	0	$\bar{1}\cdot69622$	-1693	+21	0	$\bar{1}\cdot67950$	-1651	+21	0
	-2481	+109	-3	0	-2376	+105	-3	0	-2274	+99	-3	0	-2178	+93	-2	0	-2087	+88	-2	0
	-116	0	0	0	-115	0	0	0	-115	+1	0	0	-114	+1	0	0	-113	+1	0	0
	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0
$\bar{2}\cdot7$	$\bar{1}\cdot72295$	-1714	+20	0	$\bar{1}\cdot70599$	-1674	+19	0	$\bar{1}\cdot68944$	-1636	+19	0	$\bar{1}\cdot67327$	-1599	+19	0	$\bar{1}\cdot65747$	-1562	+19	0
	-2723	+108	-2	0	-2616	+104	-2	0	-2514	+100	-2	0	-2416	+95	-2	0	-2323	+91	-3	0
	-125	0	0	0	-124	0	0	0	-124	0	0	0	-124	+1	0	0	-123	+1	0	0
	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0
$\bar{2}\cdot8$	$\bar{1}\cdot69444$	-1606	+18	0	$\bar{1}\cdot67856$	-1570	+17	0	$\bar{1}\cdot66303$	-1536	+17	0	$\bar{1}\cdot64784$	-1503	+17	0	$\bar{1}\cdot63298$	-1470	+16	0
	-2981	+109	-2	0	-2874	+105	-2	0	-2771	+100	-2	0	-2673	+97	-2	0	-2578	+93	-2	0
	-134	0	0	0	-134	0	0	0	-134	+1	0	0	-133	0	0	0	-133	0	0	0
	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0
$\bar{2}\cdot9$	$\bar{1}\cdot66326$	-1497	+16	0	$\bar{1}\cdot64845$	-1465	+15	0	$\bar{1}\cdot63395$	-1435	+15	0	$\bar{1}\cdot61975$	-1406	+15	0	$\bar{1}\cdot60584$	-1377	+14	0
	-3258	+109	-2	0	-3151	+104	-2	0	-3048	+100	-2	0	-2950	+97	-2	0	-2855	+94	-1	0
	-143	0	0	0	-143	0	0	0	-143	0	0	0	-143	0	0	0	-143	0	0	0
	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0	-3	0	0	0
$\bar{1}\cdot0$	$\bar{1}\cdot62922$	-1388	+14	0	$\bar{1}\cdot61548$	-1361	+13	0	$\bar{1}\cdot60201$	-1335	+13	0	$\bar{1}\cdot58879$	-1309	+13	0	$\bar{1}\cdot57583$	-1283	+13	0

z	$n=1\cdot5$				$n=1\cdot6$				$n=1\cdot7$				$n=1\cdot8$				$n=1\cdot9$			
$\bar{2}\cdot5$	$\bar{1}\cdot68212$	−1694	+22	0	$\bar{1}\cdot66540$	−1651	+21	0	$\bar{1}\cdot64910$	−1609	+21	0	$\bar{1}\cdot63322$	−1568	+20	0	$\bar{1}\cdot61774$	−1528	+20	0
	−1787	+83	−2	0	−1707	+78	−2	0	−1632	+74	−2	0	−1560	+69	−2	0	−1493	+65	−2	0
	−102	+1	0	0	−101	+1	0	0	−99	+1	0	0	−98	+2	0	0	−96	+2	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{2}\cdot6$	$\bar{1}\cdot66320$	−1610	+20	0	$\bar{1}\cdot64729$	−1572	+19	0	$\bar{1}\cdot63176$	−1534	+19	0	$\bar{1}\cdot61661$	−1497	+18	0	$\bar{1}\cdot60182$	−1461	+18	0
	−2000	+84	−2	0	−1918	+80	−1	0	−1840	+76	−2	0	−1766	+72	−1	0	−1695	+69	−2	0
	−112	+1	0	0	−111	+1	0	0	−109	+1	0	0	−108	+1	0	0	−107	+1	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{2}\cdot7$	$\bar{1}\cdot64204$	−1525	+18	0	$\bar{1}\cdot62697$	−1491	+18	0	$\bar{1}\cdot61224$	−1457	+17	0	$\bar{1}\cdot59784$	−1424	+17	0	$\bar{1}\cdot58377$	−1391	+16	0
	−2234	+86	−2	0	−2150	+83	−2	0	−2069	+79	−2	0	−1992	+75	−2	0	−1919	+71	−2	0
	−122	+1	0	0	−121	+1	0	0	−120	+1	0	0	−119	+1	0	0	−118	+1	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{2}\cdot8$	$\bar{1}\cdot61845$	−1438	+16	0	$\bar{1}\cdot60423$	−1407	+16	0	$\bar{1}\cdot59032$	−1377	+15	0	$\bar{1}\cdot57670$	−1348	+15	0	$\bar{1}\cdot56337$	−1319	+14	0
	−2488	+89	−2	0	−2401	+84	−2	0	−2319	+81	−2	0	−2240	+78	−2	0	−2164	+75	−1	0
	−132	0	0	0	−132	+1	0	0	−131	+1	0	0	−130	+1	0	0	−129	+1	0	0
	−4	0	0	0	−4	0	0	0	−4	0	0	0	−4	0	0	0	−4	0	0	0
$\bar{2}\cdot9$	$\bar{1}\cdot59221$	−1349	+14	0	$\bar{1}\cdot57886$	−1322	+14	0	$\bar{1}\cdot56578$	−1295	+13	0	$\bar{1}\cdot55296$	−1269	+13	0	$\bar{1}\cdot54040$	−1243	+13	0
	−2762	+89	−2	0	−2675	+86	−2	0	−2591	+83	−2	0	−2510	+79	−2	0	−2433	+75	−2	0
	−143	+1	0	0	−142	+1	0	0	−141	0	0	0	−141	+1	0	0	−140	+1	0	0
	−4	0	0	0	−4	0	0	0	−4	0	0	0	−4	0	0	0	−4	0	0	0
$\bar{1}\cdot0$	$\bar{1}\cdot56312$	−1259	+12	0	$\bar{1}\cdot55065$	−1235	+12	0	$\bar{1}\cdot53842$	−1212	+11	0	$\bar{1}\cdot52641$	−1189	+11	0	$\bar{1}\cdot51463$	−1167	+11	0

x	$n=1\cdot0$				$n=1\cdot1$				$n=1\cdot2$				$n=1\cdot3$				$n=1\cdot4$			
$\bar{1}\cdot0$	$\bar{1}\cdot62922$	−1388	+14	0	$\bar{1}\cdot61548$	−1361	+13	0	$\bar{1}\cdot60201$	−1335	+13	0	$\bar{1}\cdot58879$	−1309	+13	0	$\bar{1}\cdot57583$	−1283	+13	0
	−3554	+108	−2	0	−3447	+105	−2	0	−3345	+101	−2	0	−3246	+97	−2	0	−3151	+93	−2	0
	−152	0	0	0	−153	0	0	0	−153	0	0	0	−153	0	0	0	−153	0	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{1}\cdot1$	$\bar{1}\cdot59213$	−1280	+12	0	$\bar{1}\cdot57945$	−1256	+11	0	$\bar{1}\cdot56700$	−1234	+11	0	$\bar{1}\cdot55477$	−1212	+11	0	$\bar{1}\cdot54276$	−1190	+11	0
	−3866	+106	−1	0	−3762	+103	−1	0	−3660	+100	−1	0	−3561	+96	−2	0	−3467	+92	−2	0
	−161	−1	0	0	−161	−1	0	0	−162	−1	0	0	−163	0	0	0	−163	0	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{1}\cdot2$	$\bar{1}\cdot55183$	−1175	+11	0	$\bar{1}\cdot54019$	−1154	+10	0	$\bar{1}\cdot52875$	−1135	+10	0	$\bar{1}\cdot51750$	−1116	+9	0	$\bar{1}\cdot50643$	−1098	+9	0
	−4197	+105	−2	0	−4094	+102	−2	0	−3994	+99	−2	0	−3897	+96	−1	0	−3802	+92	−1	0
	−169	−1	0	0	−169	−1	0	0	−171	−1	0	0	−171	−1	0	0	−172	0	0	0
	−2	0	0	0	−3	0	0	0	−2	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{1}\cdot3$	$\bar{1}\cdot50815$	−1071	+9	0	$\bar{1}\cdot49753$	−1053	+8	0	$\bar{1}\cdot48708$	−1037	+8	0	$\bar{1}\cdot47679$	−1021	+8	0	$\bar{1}\cdot46666$	−1006	+8	0
	−4542	+103	−1	0	−4440	+99	−1	0	−4343	+96	−1	0	−4248	+93	−1	0	−4155	+91	−1	0
	−175	−2	0	0	−177	−2	0	0	−178	−1	0	0	−179	−1	0	0	−180	−1	0	0
	−2	0	0	0	−3	0	0	0	−2	0	0	0	−2	0	0	0	−3	0	0	0
$\bar{1}\cdot4$	$\bar{1}\cdot46096$	−970	+8	0	$\bar{1}\cdot45134$	−956	+7	0	$\bar{1}\cdot44185$	−942	+7	0	$\bar{1}\cdot43250$	−929	+7	0	$\bar{1}\cdot42328$	−916	+7	0
	−4899	+99	−2	0	−4801	+97	−2	0	−4706	+94	−2	0	−4613	+92	−2	0	−4523	+90	−2	0
	−181	−2	0	0	−183	−1	0	0	−184	−1	0	0	−186	−1	0	0	−187	−1	0	0
	−1	0	0	0	−2	0	0	0	−2	0	0	0	−2	0	0	0	−3	0	0	0
$\bar{1}\cdot5$	$\bar{1}\cdot41015$	−873	+6	0	$\bar{1}\cdot40148$	−860	+5	0	$\bar{1}\cdot39293$	−849	+5	0	$\bar{1}\cdot38449$	−838	+5	0	$\bar{1}\cdot37615$	−827	+5	0

x	$n=1\cdot5$				$n=1\cdot6$				$n=1\cdot7$				$n=1\cdot8$				$n=1\cdot9$			
$\bar{1}\cdot0$	$\bar{1}\cdot56312$	−1259	+12	0	$\bar{1}\cdot55065$	−1235	+12	0	$\bar{1}\cdot53842$	−1212	+11	0	$\bar{1}\cdot52641$	−1189	+11	0	$\bar{1}\cdot51463$	−1167	+11	0
	−3058	+90	−2	0	−2970	+87	−2	0	−2885	+83	−1	0	−2803	+80	−2	0	−2725	+76	−2	0
	−153	0	0	0	−153	0	0	0	−153	+1	0	0	−152	+1	0	0	−151	+1	0	0
	−4	0	0	0	−4	0	0	0	−4	0	0	0	−4	0	0	0	−4	0	0	0
$\bar{1}\cdot1$	$\bar{1}\cdot53097$	−1169	+10	0	$\bar{1}\cdot51938$	−1148	+10	0	$\bar{1}\cdot50800$	−1128	+10	0	$\bar{1}\cdot49682$	−1109	+10	0	$\bar{1}\cdot48583$	−1090	+9	0
	−3376	+90	−1	0	−3287	+87	−1	0	−3201	+84	−2	0	−3118	+81	−2	0	−3039	+78	−1	0
	−163	0	0	0	−163	0	0	0	−163	0	0	0	−163	+1	0	0	−162	+1	0	0
	−3	0	0	0	−3	0	0		−3	0	0	0	−4	0	0	0	−4	0	0	0
$\bar{1}\cdot2$	$\bar{1}\cdot49555$	−1079	+9	0	$\bar{1}\cdot48485$	−1061	+9	0	$\bar{1}\cdot47433$	−1044	+8	0	$\bar{1}\cdot46397$	−1027	+8	0	$\bar{1}\cdot45378$	−1011	+8	0
	−3712	+90	−1	0	−3623	+87	−1	0	−3538	+84	−1	0	−3455	+81	−1	0	−3375	+79	−1	0
	−172	−1	0	0	−173	0	0	0	−173	0	0	0	−173	0	0	0	−173	0	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{1}\cdot3$	$\bar{1}\cdot45668$	−990	+8	0	$\bar{1}\cdot44686$	−975	+8	0	$\bar{1}\cdot43719$	−960	+7	0	$\bar{1}\cdot42766$	−946	+7	0	$\bar{1}\cdot41827$	−932	+7	0
	−4065	+88	−1	0	−3978	+85	−1	0	−3894	+83	−1	0	−3812	+81	−1	0	−3732	+79	−1	0
	−181	−1	0	0	−182	0	0	0	−182	−1	0	0	−183	0	0	0	−183	0	0	0
	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0	−3	0	0	0
$\bar{1}\cdot4$	$\bar{1}\cdot41419$	−903	+7	0	$\bar{1}\cdot40523$	−890	+7	0	$\bar{1}\cdot39640$	−878	+6	0	$\bar{1}\cdot38768$	−866	+6	0	$\bar{1}\cdot37909$	−853	+6	0
	−4436	+87	−2	0	−4351	+84	−2	0	−4268	+82	−1	0	−4187	+80	−1	0	−4108	+78	−1	0
	−188	−1	0	0	−189	−1	0	0	−190	0	0	0	−190	−1	0	0	−191	−1	0	0
	−2	0	0	0	−2	0	0	0	−2	0	0	0	−2	0	0	0	−2	0	0	0
$\bar{1}\cdot5$	$\bar{1}\cdot36793$	−817	+5	0	$\bar{1}\cdot35981$	−806	+5	0	$\bar{1}\cdot35180$	−796	+5	0	$\bar{1}\cdot34389$	−786	+5	0	$\bar{1}\cdot33608$	−776	+5	0

z	$n=1·0$				$n=1·1$				$n=1·2$				$n=1·3$				$n=1·4$			
$\bar{1}$·5	$\bar{1}$·41015 −5266 −185 −1	−873 +99 −3 0	+6 −1 0 0	0 0 0 0	$\bar{1}$·40148 −5174 −186 −1	−860 +93 −2 0	+5 −1 0 0	0 0 0 0	$\bar{1}$·39293 −5081 −189 −1	−849 +90 −2 0	+5 −1 0 0	0 0 0 0	$\bar{1}$·38449 −4992 −191 −1	−838 +88 −2 0	+5 −1 0 0	0 0 0 0	$\bar{1}$·37615 −4905 −192 −1	−827 +86 −2 0	+5 −1 0 0	0 0 0 0
$\bar{1}$·6	$\bar{1}$·35563 −5641 −188 0	−781 +92 −3 0	+5 −1 0 0	0 0 0 0	$\bar{1}$·34787 −5552 −189 0	−769 +88 −3 0	+4 −1 0 0	0 0 0 0	$\bar{1}$·34022 −5464 −192 0	−761 +86 −2 0	+4 −1 0 0	0 0 0 0	$\bar{1}$·33265 −5379 −194 0	−752 +84 −2 0	+4 0 0 0	0 0 0 0	$\bar{1}$·32517 −5294 −196 0	−743 +83 −2 0	+4 0 0 0	0 0 0 0
$\bar{1}$·7	$\bar{1}$·29734 −6016 −188 0	−692 +86 −3 0	+4 −1 0 0	0 0 0 0	$\bar{1}$·29046 −5932 −190 0	−684 +83 −2 0	+3 −1 0 0	0 0 0 0	$\bar{1}$·28366 −5849 −193 0	−677 +82 −2 0	+3 0 0 0	0 0 0 0	$\bar{1}$·27692 −5768 −195 0	−670 +80 −2 0	+4 −1 0 0	0 0 0 0	$\bar{1}$·27026 −5688 −197 0	−662 +78 −2 0	+4 −1 0 0	0 0 0 0
$\bar{1}$·8	$\bar{1}$·23530 −6392 −186 +1	−609 +80 −3 0	+3 −1 0 0	0 0 0 0	$\bar{1}$·22924 −6314 −188 +1	−603 +78 −2 0	+2 −0 0 0	0 0 0 0	$\bar{1}$·22323 −6235 −191 +1	−597 +77 −3 0	+3 −1 0 0	0 0 0 0	$\bar{1}$·21729 −6157 −194 +1	−592 +76 −2 0	+3 −1 0 0	0 0 0 0	$\bar{1}$·21140 −6082 −196 +1	−586 +74 −2 0	+3 −1 0 0	0 0 0 0
$\bar{1}$·9	$\bar{1}$·16953 −6760 −182 +2	−532 +74 −3 0	+2 −1 0 0	0 0 0 0	$\bar{1}$·16423 −6686 −185 +1	−527 +72 −3 0	+2 −1 0 0	0 0 0 0	$\bar{1}$·15898 −6614 −189 +1	−523 +71 −2 0	+2 0 0 0	0 0 0 0	$\bar{1}$·15377 −6542 −192 +1	−518 +71 −3 0	+2 −1 0 0	0 0 0 0	$\bar{1}$·14861 −6472 −194 +1	−514 +69 −2 0	+2 −1 0 0	0 0 0 0
·0	$\bar{1}$·10013	−461	+1	0	$\bar{1}$·09553	−458	+1	0	$\bar{1}$·09096	−454	+2	0	$\bar{1}$·08644	−450	+1	0	$\bar{1}$·08195	−447	+1	0

z	$n=1\cdot5$				$n=1\cdot6$				$n=1\cdot7$				$n=1\cdot8$				$n=1\cdot9$			
$\bar{1}\cdot5$	$1\cdot36793$	−817	+5	0	$\bar{1}\cdot35981$	−806	+5	0	$\bar{1}\cdot35180$	−796	+5	0	$\bar{1}\cdot34389$	−786	+5	0	$\bar{1}\cdot33608$	−776	+5	0
	−4819	+84	−1	0	−4736	+82	−1	0	−4655	+80	−1	0	−4576	+78	−1	0	−4499	+76	−1	0
	−194	−1	0	0	−195	−1	0	0	−196	−1	0	0	−197	−1	0	0	−198	−1	0	0
	−2	0	0	0	−2	0	0	0	−2	0	0	0	−2	0	0	0	−2	0	0	0
$\bar{1}\cdot6$	$\bar{1}\cdot31778$	−734	+4	0	$\bar{1}\cdot31048$	−725	+4	0	$\bar{1}\cdot30327$	−717	+4	0	$\bar{1}\cdot29614$	−709	+4	0	$\bar{1}\cdot28909$	−701	+4	0
	−5212	+81	−1	0	−5132	+79	−1	0	−5054	+77	−1	0	−4978	+76	−1	0	−4903	+75	0	0
	−197	−2	0	0	−199	−2	0	0	−201	−1	0	0	−202	−1	0	0	−203	−1	0	0
	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0	−1	0	0	0
$\bar{1}\cdot7$	$\bar{1}\cdot26368$	−655	+3	0	$\bar{1}\cdot25716$	−648	+3	0	$\bar{1}\cdot25071$	−642	+4	0	$\bar{1}\cdot24433$	−634	+3	0	$\bar{1}\cdot23802$	−627	+4	0
	−5611	+77	0	0	−5534	+76	−1	0	−5459	+74	−1	0	−5386	+73	0	0	−5313	+72	−1	0
	−199	−2	0	0	−201	−2	0	0	−203	−1	0	0	−204	−2	0	0	−206	−2	0	0
	−0	0	0	0	−0	0	0	0	−0	0	0	0	−0	0	0	0	−0	0	0	0
$\bar{1}\cdot8$	$\bar{1}\cdot20557$	−580	+3	0	$\bar{1}\cdot19980$	−574	+2	0	$\bar{1}\cdot19408$	−569	+3	0	$\bar{1}\cdot18842$	−563	+3	0	$\bar{1}\cdot18282$	−558	+3	0
	−6009	+73	−1	0	−5937	+71	0	0	−5866	+70	0	0	−5796	+69	−1	0	−5728	+68	−1	0
	−199	−2	0	0	−201	−2	0	0	−203	−2	0	0	−205	−2	0	0	−207	−2	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−0	0	0	0
$\bar{1}\cdot9$	$\bar{1}\cdot14349$	−509	+2	0	$\bar{1}\cdot13842$	−505	+2	0	$\bar{1}\cdot13339$	−501	+3	0	$\bar{1}\cdot12841$	−497	+2	0	$\bar{1}\cdot12346$	−493	+2	0
	−6404	+67	0	0	−6336	+67	0	0	−6269	+66	−1	0	−6205	+65	0	0	−6140	+64	−1	0
	−198	−2	0	0	−200	−2	0	0	−202	−2	0	0	−204	−2	0	0	−206	−2	0	0
	+1	0	0	0	+1	0	0	0	+1	0	0	0	+1	0	0	0	+1	0	0	0
·0	$\bar{1}\cdot07749$	−444	+2	0	$\bar{1}\cdot07307$	−440	+2	0	$\bar{1}\cdot06869$	−437	+2	0	$\bar{1}\cdot06433$	−434	+2	0	$\bar{1}\cdot06001$	−431	+1	0

Notes on the Observations of the Rev. John Hodgson, M.A., on the Mortality of the Clergy of England and Wales, with Remarks on the Tables deduced therefrom by Mr. Samuel Brown. By WILFRED ARTHUR BOWSER, *Actuary and Secretary of the London and County Provident Institution.*

IN the year 1865, the late Rev. John Hodgson published a pamphlet entitled "*Observations in reference to the Duration of Life amongst the Clergy of England and Wales;*" to which is added, in the form of a supplement, a letter from Mr. Samuel Brown, containing various tables, having the observations of Mr. Hodgson for their basis, applicable to questions connected with Life Assurance, and the Valuation of Next Presentations to Livings and Advowsons.

The original observations embrace 5088 entrants, being mostly incumbents of livings, upwards of 100,000 years of life, and 3122 deaths. The extent of the experience, combined with the fact that it extends over a century, and the careful manner in which the statistics were compiled, impart confidence to the observations considered as an index of the value of clergy life.

Mr. Brown has incidentally touched upon the comparative mortality amongst the total entrants and the entrants during this century only, but has not given comparisons between entrants this and last century. Mr. Hodgson's Table No. 3, however, enables us to extract these particulars, as it shows, in quinquennial divisions, the number of entrants, the number under observation, and the number of deaths, both in respect of entrants this century and of the combined data. By subtracting the former from the latter the data for the entrants last century may be obtained; I therefore give the comparisons in the following table:—

Mortality amongst the Clergy of England and Wales from 1760 *to* 1860, *distinguishing Entrants this Century from Entrants last Century.*

Ages.	Force of Mortality percent amongst			Number of Deaths upon which the Percentages are computed.		
	Entrants last Century.	Entrants this Century.	All Entrants.	Entrants last Century.	Entrants this Century.	All Entrants.
24–29	·501	·422	·473	11	5	16
30–34	·643	·635	·639	25	17	42
35–39	·771	·620	·706	43	26	69
40–44	·942	·729	·846	63	40	103
45–49	1·209	·835	1·035	90	54	144
50–54	1·998	1·333	1·686*	153	90	243
55–59	2·814	1·489*	2·213	205	90	295
60–64	3·511	2·910	3·255	230	141	371
65–69	4·726	4·396	4·592	255	161	416
70–74	7·683	6·801	7·336*	301	172	473
75–79	12·184	10·724	11·626	289	157	446
80–84	17·140	16·729*	17·005	193	109	302
85–100	26·131	23·934	25·284	127	73	200
101–102	...	...	...	...	2	2
24–102	3·276	2·457	2·922	1985	1137	3122

There is a marked diminution in the mortality among entrants during this century, at all ages, but chiefly noticeable at the prime of life, viz., between ages 45 and 60, where the diminution amounts to about 32 percent.

Mr. Hodgson's "numbers rated to be under observation" are arrived at in a different manner to that usually adopted, and are such that, if divided into the number of deaths, the "force of mortality," and not the "probability of dying," will be (approximately) obtained. In constructing the accompanying tables from Mr. Hodgson's original observations, I have therefore disregarded his "numbers rated to be under observation," and have obtained the required number at each age by adding to the continuants from the preceding age half the entrants, and subtracting half the discontinuants *otherwise than by death.* For the purpose of comparison and verification, Mr. Hodgson's Table No. 1 is here reprinted in full, the last column having been added by me to show the numbers subject to mortality when calculated in the ordinary way.

* Incorrectly given as 1·700, 1·655, 7·525, 16·770 respectively in Mr. Hodgson's tables. Several other percentages are slightly incorrect in the last decimal place.

" *Table No.* 1, *showing at what Ages* 5088 *Incumbents, &c.* (*included* " *in groups* 1 *to* 8) *entered under Observation* (*Column* 3); *at* " *what Ages* 3122 *of them were taken out of Observation by Death* " (*Column* 5); *and at what Ages the remaining* 1966 *were taken* " *out of Observation otherwise than by Death* (*Column* 4); *and* " *also the Number of them which were rated under Observation at* " *every Age from* 24 *to* 101 (*Column* 6)."

1. Age arrived at.	2. Continuants at the Age.	3. Entrants during the Year of Age.	4. Taken out of Observation during the Year of Age otherwise than by Death.	5. Died during the Year of Age.	6. Number Rated to be under Observation during the Year of Age.	7. Corrected Number subject to Mortality.
24	0	279	0	1	167	139·5
25	278	160	2	1	362	357·
26	435	141	5	2	509	503·
27	569	126	8	4	630	628·
28	683	180	7	1	778	769·5
29	855	165	10	7	932	932·5
30	1003	143	17	4	1070	1066·
31	1125	154	15	8	1199	1194·5
32	1256	148	19	5	1314	1320·5
33	1380	140	17	14	1430	1441·5
34	1489	147	13	11	1554	1556·
35	1612	180	16	8	1709	1694·
36	1768	169	20	12	1837	1842·5
37	1905	154	23	16	1965	1970·5
38	2020	148	24	11	2083	2082·
39	2133	132	26	22	2180	2186·
40	2217	138	26	12	2275	2273·
41	2317	125	24	27	2357	2367·5
42	2391	127	27	17	2439	2441·
43	2474	143	39	24	2517	2526·
44	2554	127	37	23	2584	2599·
45	2621	133	29	24	2662	2673·
46	2701	113	31	20	2739	2742·
47	2763	118	39	33	2789	2802·5
48	2809	123	37	28	2839	2852·
49	2867	91	38	39	2881	2893·5
50	2881	123	48	40	2907	2918·5
51	2916	91	52	57	2909	2935·5
52	2898	84	52	51	2890	2914·
53	2879	82	55	54	2859	2892·5
54	2852	78	63	41	2847	2859·5
55	2826	66	66	62	2791	2826·
56	2764	63	73	57	2727	2759·
57	2697	67	67	54	2673	2697·
58	2643	46	63	56	2606	2634·5
59	2570	48	60	66	2529	2564·
60	2492	44	68	64	2440	2480·
61	2404	59	55	69	2372	2406·
62	2339	34	54	71	2291	2329·
63	2248	39	50	86	2194	2242·5
64	2151	49	52	81	2100	2149·5
65	2067	30	51	72	2020	2056·5
66	1974	30	45	99	1909	1966·5
67	1860	24	45	77	1810	1849·5
68	1762	27	41	86	1706	1755·
69	1662	15	33	82	1613	1653·
70	1562	27	39	82	1512	1556·

" *Table No.* 1—(continued).

1. Age arrived at.	2. Continuants at the Age.	3. Entrants during the Year of Age.	4. Taken out of Observation during the Year of Age otherwise than by Death.	5. Died during the Year of Age.	6. Number Rated to be under Observation during the Year of Age.	7. Corrected Number subject to Mortality.
71	1468	15	27	100	1404	1462·
72	1356	20	40	102	1287	1346·
73	1234	11	20	95	1178	1229·5
74	1130	15	25	94	1066	1125·
75	1026	17	27	99	966	1021·
76	917	14	33	94	856	907·5
77	804	19	20	75	763	803·5
78	728	10	15	96	672	725·5
79	627	4	10	82	579	624·
80	539	5	14	76	489	534·5
81	454	4	5	60	415	453·5
82	393	4	10	66	352	390·
83	321	2	8	52	285	318·
84	263	5	7	48	235	262·
85	213	4	4	36	191	213·
86	177	0	6	42	149	174·
87	129	5	3	19	119	130·
88	112	1	2	27	100	111·5
89	84	1	3	22	67	83·
90	60	1	2	12	53	59·5
91	46	1	1	17	36	46·
92	30	...	1	6	26	29·5
93	23	...	1	9	16	22·5
94	13	...	...	4	10	13·
95	9	...	...	3	7	9·
96	6	...	...	1	6	6·
97	5	...	1	0	5	4·5
98	4	...	...	0	4	4·
99	4	...	...	2	2	4·
100	2	...	...	1	1	2·
101	1	...	...	1	0	1·
102	0	...	...			

It is to be noticed that, with a view to make practical use of Mr. Hodgson's observations, Mr. Brown first summarizes the data into decennial groups in the following table, numbered 10 in the pamphlet:—

Ages.	Continuants.	Entrants.	Withdrawn from Observation.	Numbers observed.	Died.	Rate of Mortality percent.
24–35	9073	1783	113	9945	58	·583
35–45	21391	1443	262	21946	172	·784
45–55	28187	1036	444	28322	387	1·366
55–65	25134	515	608	24723	666	2·694
65–75	16075	214	366	15505	889	5·734
75–85	6072	84	149	5612	748	13·328
85–95	887	13	23	767	194	25·293
95–102	31	0	1	25	8	31·993
	106850	5088	1966	106845	3122	2·922

He then proceeds to reduce the *average* "force of mortality" *for decennial ages,* so obtained, into the average "probability of dying." He says (altering his notation to that now recommended by the Institute), "If q_x be the probability of dying in a year at "the age x, and μ_x the rate of mortality [meaning *force* of mor- "tality] given above, the former may be deduced from the latter by "the following formula:—

$$\text{“}\, q_x=\frac{2\mu_x}{2+\mu_x}.$$

"This I have done for the mean of each decennial group of ages." Upon the assumption of equal decrements throughout the year, this formula is approximately true, as applied to the force of mortality at each *separate* age: for, omitting the subscript x, we may say that

$$\mu=\frac{d}{l-\frac{d}{2}}=\frac{q}{1-\frac{q}{2}}$$

$$\therefore \quad \mu-\frac{\mu q}{2}=q$$

$$\therefore \quad 2\mu-\mu q=q$$

$$\therefore \quad 2\mu=2q+\mu q=q(2+\mu)$$

$$\therefore \quad q=\frac{2\mu}{2+\mu}.*$$

* It may be useful to point out that there is a slight inaccuracy in this formula, which will perhaps be more clearly perceived by arranging the proof in the following manner.

We have $$q_x=\frac{l_x-l_{x+1}}{l_x}=1-p_x.$$

Again, just in the same way as we have $\mu_x=\frac{l_{x-1}-l_{x+1}}{2l_x}$ approximately, we have also approximately,

$$\begin{aligned}\mu_{x+\frac{1}{2}}&=\frac{l_x-l_{x+1}}{l_{x+\frac{1}{2}}}\\&=\frac{2(l_x-l_{x+1})}{l_x+l_{x+1}},\text{ nearly,}\\&=\frac{2(1-p_x)}{1+p_x}\\&=\frac{2q_x}{2-q_x};\end{aligned}$$

whence we readily get $$q_x=\frac{2\mu_{x+\frac{1}{2}}}{2+\mu_{x+\frac{1}{2}}}.$$

It is thus seen that the μ used in calculating the value of q, must

But it is to be observed that to assume that the average probability of dying for decennial groups of ages, represents the probability of dying at the "mean" age, involves a rather serious error, there being no mean of an *even* number of terms. For instance, the arithmetical mean of ten ages, say x to $x+9$ inclusive, is evidently midway between $x+4$ and $x+5$, *i.e.* $x+4\frac{1}{2}$, whereas Mr. Brown has taken it to be $x+5$. Thus the mean of ages 35 and 44 inclusive, although properly midway between 39 and 40, say $39\frac{1}{2}$, has been taken to be 40. No allowance has been made for this assumption; whence it follows—supposing the interpolation of the intermediate ages to be properly effected, and that the average of a series of values represents the value of the middle term—that the mortality at every age is understated, more nearly representing the probability of dying in a year at ages respectively half a year younger throughout. It is chiefly to this error, which increases considerably as age advances, that the difference between the expectations of life by Mr. Brown's adjusted table and those deduced by me from the original facts, without any adjustment, is attributable; the difference amounting, on the average, to about half a year.

The method of interpolation employed by Mr. Brown for supplying the probabilities of dying at intermediate ages, stated by him to be by the "even differences" of the logarithms at every tenth age, is no doubt very simple and easy of application; but would not, I feel sure, be advocated by him upon scientific considerations. A close adherence with the results yielded by the original data is of the utmost importance in graduating any mortality table, and as Mr. Brown's adjusted table based upon Mr. Hodgson's observations unduly increases the vitality, especially at the older ages, to a somewhat serious extent, I have thought it advisable to compute the actual observed probability of dying at each separate age from the youngest to the oldest, from which I have constructed a new mortality table, and have graduated the

be that for the middle of the year of age to which q relates, and not for the beginning. It is to be noticed that μ_x relates to a point of time, the exact age x, whereas q_x relates to the year of age extending from the age x to the age $x+1$. (The student may consult with advantage Mr. Makeham's letter in vol. xiv, p. 243.) Mr. Brown's formula in the passage quoted is equivalent to $q_x = \frac{2\mu}{2+\mu}$, no subscript having been added by him to the letter used instead of μ; but his letter printed below shows that he has fallen into no error on the point. It also appears to explain fully the cause of the difference of half a year between his results and Mr. Bowser's.—Ed. *J. I. A.*

same by Mr. Woolhouse's method (*Assurance Magazine*, vol. xv page 389). The following table shows the "expectations of life" at quinquennial ages deduced from the original data without any adjustment, compared with those resulting from Mr. Woolhouse and Mr. Brown's adjustments respectively, together with the differences from the original in each case: also the values of annuities at quinquennial ages by the two methods of adjustment.

Rev. John Hodgson's Clergy Observations.
Expectation of Life.

Age.	Unadjusted.	Adjusted (Present paper).	Adjusted (Mr. Brown).	Difference from Unadjusted.	
				Present Paper.	Mr. Brown.
24	41·46	41·47	41·62	+·01	+·16
30	36·60	36·55	36·86	−·05	+·26
35	32·67	32·61	32·94	−·06	+·27
40	28·73	28·71	29·05	−·02	+·32
45	24·86	24·83	25·23	−·03	+·37
50	21·03	21·06	21·58	+·03	+·55
55	17·66	17·71	18·13	+·05	+·47
60	14·44	14·44	14·94	æq	+·50
65	11·54	11·52	12·05	−·02	+·51
70	8·88	8·88	9·50	æq	+·62
75	6·70	6·69	7·34	−·01	+·64
80	5·13	5·11	5·66	−·02	+·53
85	3·97	3·94	4·46	−·03	+·49
90	3·06	2·97	3·62	−·09	+·56
			Average	−·02	+·44

Values of Annuities. Interest 3 percent.

Age.	Present paper.	Mr. Brown.	Age.	Present paper.	Mr. Brown.
24	22·156	22·124	60	10·615	10·887
30	20·649	20·705	65	8·760	9·075
35	19·306	19·376	70	6·925	7·347
40	17·805	17·890	75	5·285	5·767
45	16·126	16·256	80	4·045	4·474
50	14·286	14·525	85	3·091	3·509
55	12·526	12·718	90	2·244	2·808

I have not thought it necessary to give tables of the values of advowsons and next presentations, as they vary considerably according to circumstances beyond the actuary's power of calculation, and the value of such property will probably show greater fluctuation in the future than in the past. It may, however, be stated that, if valued by the tables given at the end of this paper, they are worth more than if valued by Mr. Brown's tables, for the reasons already given. For example, interest being calcu-

lated at 5 percent, the value of £100 nett annual income is as follows, for the ages specified:—

Age of present Incumbent.	Advowson.		Next Presentation.	
	Present paper.	Mr. Brown.	Present paper.	Mr. Brown.
	£	£	£	£
50	815·4	800·6	661·2	647·3
55	932·2	921·0	755·9	744·7
60	1067·6	1049·5	865·7	848·6
65	1205·6	1183·2	977·5	956·8
70	1349·1	1316·7	1093·9	1064·7
75	1483·4	1444·1	1202·8	1167·7
80	1587·8	1552·0	1287·4	1255·0
85	1667·8	1634·4	1352·3	1321·6
90	1745·6	1695·4	1415·4	1370·9

Annexed will be found complete tables of the values of annuities and reversions at 3 and 5 percent interest, and of annual premiums at 3 percent. A comparison of these with Mr. Brown's tables will, I think, show that sufficient reason exists for my having called attention to the subject.

Table I, *showing the Probabilities of Dying, also the Numbers-Living and Decrement, according to the Observations by the late Rev. John Hodgson, M.A., on* 5088 *Clergymen, Incumbents of Livings, &c., in England and Wales, of whom* 3122 *had Died.*

x	Unadjusted.			Adjusted.		
	Probability of Dying.	Number-Living.	Decrement.	Number-Living.	Decrement.	Probability of Dying.
24	·00717	100000	717	100000	374	·00374
25	·00280	99283	279	99626	411	·00413
26	·00397	99004	394	99215	445	·00449
27	·00637	98610	630	98770	477	·00483
28	·00130	97980	128	98293	506	·00515
29	·00751	97852	734	97787	531	·00543
30	·00375	97118	364	97256	554	·00570
31	·00670	96754	648	96702	574	·00594
32	·00379	96106	365	96128	591	·00615
33	·00971	95741	930	95537	613	·00642
34	·00707	94811	670	94924	611	·00644
35	·00472	94141	445	94313	658	·00698
36	·00651	93696	611	93655	617	·00659
37	·00812	93085	756	93038	653	·00702
38	·00528	92329	487	92385	665	·00720
39	·01009	91842	928	91720	720	·00785
40	·00528	90914	480	91000	714	·00784
41	·01140	90434	1032	90286	764	·00846
42	·00696	89402	621	89522	758	·00847
43	·00950	88781	844	88764	779	·00878
44	·00885	87937	779	87985	742	·00843
45	·00898	87158	783	87243	787	·00902
46	·00729	86375	630	86456	806	·00932
47	·01177	85745	1009	85650	892	·01041

Table I—(continued).

x	Unadjusted. Probability of Dying.	Unadjusted. Number-Living.	Unadjusted. Decre-ment.	Adjusted. Number-Living.	Adjusted. Decre-ment.	Adjusted. Probability of Dying.
48	·00982	84736	833	84758	966	·01140
49	·01348	83903	1132	83792	1116	·01338
50	·01371	82771	1136	82676	1213	·01467
51	·01942	81635	1585	81463	1323	·01624
52	·01750	80050	1402	80140	1360	·01697
53	·01867	78648	1469	78780	1432	·01817
54	·01434	77179	1106	77348	1441	·01863
55	·02194	76073	1670	75907	1463	·01927
56	·02066	74403	1538	74444	1474	·01980
57	·02002	72865	1458	72970	1557	·02133
58	·02126	71407	1518	71413	1611	·02256
59	·02574	69889	1800	69802	1690	·02421
60	·02580	68089	1756	68112	1803	·02647
61	·02868	66333	1903	66309	1947	·02936
62	·03049	64430	1965	64362	2049	·03183
63	·03835	62465	2395	62313	2146	·03444
64	·03768	60070	2263	60167	2265	·03764
65	·03501	57807	2024	57902	2338	·04038
66	·05034	55783	2808	55564	2349	·04227
67	·04163	52975	2206	53215	2389	·04489
68	·04900	50769	2488	50826	2449	·04818
69	·04961	48281	2395	48377	2505	·05178
70	·05270	45886	2418	45872	2574	·05611
71	·06840	43468	2973	43298	2744	·06337
72	·07578	40495	2198	40554	2855	·07040
73	·07727	38297	3764	37699	2955	·07838
74	·08355	34533	2885	34744	3007	·08655
75	·09696	31648	3068	31737	3017	·09506
76	·10358	28580	2961	28720	2939	·10233
77	·09334	25619	2391	25781	2863	·11105
78	·13233	23228	3074	22918	2740	·11955
79	·13141	20154	2649	20178	2570	·12736
80	·14219	17505	2489	17608	2443	·13874
81	·13230	15016	1987	15165	2241	·14777
82	·16923	13029	2205	12924	2031	·15715
83	·16352	10824	1770	10893	1809	·16607
84	·18321	9054	1659	9084	1635	·17999
85	·16901	7395	1250	7449	1377	·18485
86	·24138	6145	1483	6072	1200	·19763
87	·14615	4662	681	4872	1026	·21059
88	·24215	3981	964	3846	865	·22491
89	·26505	3017	800	2981	705	·23650
90	·20168	2217	447	2276	597	·26230
91	·36956	1770	654	1679	477	·28410
92	·20339	1116	227	1202	362	·30117
93	·40000	889	356	840	274	·32619
94	·30769	533	164	566	183	·32332
95	·33333	369	123	383	118	·30809
96	·16667	246	41	265	65	·24528
97	·00000	205	0	200	46	·23000
98	·00000	205	0	154	30	·19480
99	·50000	205	103	124	32	·25806
100	·50000	102	51	92	30	·32608
101	1·00000	51	51	62	31	·50000
102	...	0	...	31	21	·67742
103	...	...	...	10	10	1·00000

Table II.—*Values of Annuities on Single Lives at 3 and 5 percent Interest. (First payment due at end of the Year.)*

Age.	3 percent.	5 percent.	Age.	3 percent.	5 percent.
24	22·1565	16·2172	64	9·1188	7·9280
25	21·9069	16·0920	65	8·7598	7·6500
26	21·6576	15·9666	66	8·4022	7·3705
27	21·4078	15·8405	67	8·0363	7·0807
28	21·1570	15·7132	68	7·6665	6·7842
29	20·9045	15·5843	69	7·2962	6·4840
30	20·6492	15·4528	70	6·9255	6·1800
31	20·3905	15·3184	71	6·5573	5·8747
32	20·1277	15·1804	72	6·2110	5·5859
33	19·8597	15·0380	73	5·8818	5·3093
34	19·5876	14·8919	74	5·5735	5·0489
35	19·3060	14·7378	75	5·2847	4·8037
36	19·0248	14·5834	76	5·0150	4·5737
37	18·7255	14·4141	77	4·7543	4·3499
38	18·4236	14·2418	78	4·5087	4·1380
39	18·1139	14·0623	79	4·2746	3·9348
40	17·8052	13·8824	80	4·0455	3·7346
41	17·4845	13·6918	81	3·8381	3·5531
42	17·1627	13·4992	82	3·6387	3·3776
43	16·8284	13·2950	83	3·4466	3·2078
44	16·4867	13·0833	84	3·2570	3·0389
45	16·1263	12·8545	85	3·0910	2·8912
46	15·7607	12·6200	86	2·9058	2·7242
47	15·3863	12·3757	87	2·7301	2·5650
48	15·0147	12·1312	88	2·5622	2·4117
49	14·6434	11·8846	89	2·4049	2·2670
50	14·2863	11·6473	90	2·2443	2·1177
51	13·9340	11·4118	91	2·1336	2·0143
52	13·5889	11·1802	92	2·0697	1·9543
53	13·2382	10·9418	93	2·0504	1·9364
54	12·8878	10·7016	94	2·1343	2·0174
55	12·5265	10·4500	95	2·2488	2·1305
56	12·1558	10·1881	96	2·3466	2·2331
57	11·7734	9·9137	97	2·2039	2·1068
58	11·3910	9·6363	98	1·9481	1·8730
59	11·0035	9·3517	99	1·4920	1·4424
60	10·6148	9·0629	100	1·0713	1·0413
61	10·2306	8·7747	101	·6375	·6225
62	9·8562	8·4922	102	·3132	·3072
63	9·4857	8·2100	103	·0000	·0000

Table III.—*Present Values of an Unit payable at the end of the Year in which a Single Life may fail, at 3 and 5 percent Interest.*

Age.	3 percent.	5 percent.	Age.	3 percent.	5 percent.
24	·32555	·18014	32	·38463	·22950
25	·33281	·18609	33	·39244	·23629
26	·34007	·19207	34	·40036	·24324
27	·34735	·19808	35	·40857	·25058
28	·35465	·20414	36	·41675	·25794
29	·36201	·21028	37	·42547	·26600
30	·36945	·21653	38	·43427	·27420
31	·37698	·22293	39	·44329	·28275

TABLE III—(continued).

Age.	3 percent.	5 percent.	Age.	3 percent.	5 percent.
40	·45228	·29131	72	·78997	·68639
41	·46163	·30039	73	·79956	·69955
42	·47100	·30956	74	·80854	·71196
43	·48074	·31928	75	·81685	·72364
44	·49069	·32937	76	·82480	·73459
45	·50118	·34027	77	·83239	·74524
46	·51182	·35143	78	·83956	·75533
47	·52278	·36306	79	·84637	·76501
48	·53355	·37470	80	·85304	·77454
49	·54437	·38645	81	·85909	·78319
50	·55477	·39776	82	·86490	·79154
51	·56503	·40896	83	·87050	·79963
52	·57509	·41999	84	·87601	·80768
53	·58530	·43134	85	·88084	·81470
54	·59551	·44278	86	·88624	·82266
55	·60603	·45476	87	·89136	·83024
56	·61683	·46724	88	·89625	·83754
57	·62796	·48011	89	·90082	·84443
58	·63910	·49351	90	·90550	·85154
59	·65039	·50706	91	·90873	·85647
60	·66170	·52081	92	·91059	·85932
61	·67290	·53454	93	·91116	·86017
62	·68381	·54799	94	·90870	·85632
63	·69460	·56143	95	·90538	·85093
64	·70528	·57486	96	·90254	·84605
65	·71573	·58810	97	·90669	·85205
66	·72615	·60141	98	·91414	·86319
67	·73681	·61521	99	·92742	·88369
68	·74759	·62932	100	·93967	·90280
69	·75837	·64362	101	·95231	·92274
70	·76916	·65810	102	·96174	·93776
71	·77989	·67264	103	·97087	·95238

TABLE IV.—*Annual Premium for the Assurance of an Unit payable at the end of the Year in which a Single Life may fail, at 3 per-cent Interest.*

Age.	Annual Premium.	Age.	Annual Premium.	Age.	Annual Premium.
24	·014058	40	·024051	56	·046886
25	·014529	41	·024973	57	·049162
26	·015009	42	·025932	58	·051578
27	·015501	43	·026965	59	·054183
28	·016006	44	·028060	60	·056971
29	·016526	45	·029263	61	·059917
30	·017065	46	·030537	62	·062987
31	·017624	47	·031900	63	·066242
32	·018205	48	·033317	64	·069700
33	·018813	49	·034799	65	·073335
34	·019447	50	·036292	66	·077232
35	·020120	51	·037835	67	·081539
36	·020812	52	·039419	68	·086261
37	·021570	53	·041107	69	·091411
38	·022358	54	·042879	70	·097049
39	·023192	55	·044803		

[We thought it right to communicate the above paper to Mr. Brown, and he has favoured us with the following letter on the subject.—Ed. *J. I. A.*]

REV. JOHN HODGSON'S CLERGY LIFE TABLES.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—In reply to Mr. Bowser's remarks on the above Tables, I beg to offer a few words of explanation.

1. That in Table 1, Col. 6, "Number rated to be under Observation during the year of age," Mr. Hodgson appears to have taken great pains to ensure accuracy—even tracing the Lives by months of observation. Mr. Bowser has employed the usual method, which we adopt when the facts are not so minutely given; but I think it would be nearer the truth to add half the number of deaths to the number in Col. 6 for the corrected number subject to mortality at the beginning of the year of age.

2. The force of mortality computed from the average of the decennial groups of ages, would very nearly represent the force at the exact ages 30, 40, 50, ..., and the "probability of dying in the year" deduced by the formula, would consequently be for the age $29\frac{1}{2}$ to $30\frac{1}{2}$, $39\frac{1}{2}$ to $40\frac{1}{2}$, ... I was induced to put it on to the completed ages 30, 40, 50, ... without further correction, in consideration of the evident tendency in the original observations to show a diminishing rate of mortality in the present century. To keep strictly to the whole observations for 1760 to 1860, it would no doubt have been more correct to have taken the average of the two successive ages $29\frac{1}{2}$ to $31\frac{1}{2}$, ...

3. In regard to my rough method of graduation, I have found that for official purposes it makes a better table for use than more scientific methods frequently do, as any one may discover who tries to use the new Tables of the Institute for Short Periods and Survivorships. I have always regretted that Mr. Makeham's method was not adopted for these Tables, likely to be so extensively employed in office calculations. My plan follows the fluctuations of the original Table of Observations, and makes use of all the facts to obtain the averages. It merely assumes for the intervals, what nearly all Mortality Tables seem to indicate, that under ordinary circumstances the rate of mortality increases or diminishes by geometrical progression.

I am, Sir, your obedient Servant,

SAMUEL BROWN.

11 *Lombard Street,*
17 *March* 1873.

On the application of the Binomial Law to Statistical Enquiries, illustrated by the Law of the growth of Man at different ages. *By* Samuel Brown, F.S.S.

[Read before the Institute, 24 February 1873.]

In recent years, the writer who has done the most good service in bringing scientific methods to bear on the collection and comparison of statistics, is M. Quetelet, the Director of the Royal Observatory in Bruxelles, President of the Central Commission of Statistics, and the Perpetual Secretary of the Academy of Sciences. Nearly all the learned Academies of Europe claim the credit of enrolling him as a member. He was amongst the small number of illustrious men, including Professor Babbage, Whewell, Malthus, Drinkwater, Jones, Col. Sykes, &c., who at the third meeting of the British Association for the advancement of Science, at Cambridge in 1833, succeeded in establishing the special Section F on Economic Science and Statistics, and which afterwards led to the formation of the Statistical Society of London and various provincial Societies. He was also the originator of the International Statistical Congresses, which have been so useful in improving the methods of collection and publication of facts in all branches of statistical enquiry by every Government in Europe.

As far back as 1835, M. Quetelet, in his work entitled "*Sur l'homme et sur le développement de ses facultés, ou Essai de Physique sociale,*" published a series of researches of a most interesting character into the physical and moral condition of man. This excellent work, enriched with additional observations and Tables, the results of 35 years' further experience, and the zealous aid of scientific friends, was republished in 1869, in compliment to the Delegates of the Continental Governments about to assemble at the Hague, who were then and are now engaged in preparing a systematic comparison of international Statistics in every branch which affects the condition or progress of the people.

In treating of the physical qualities of man, the first and most essential part of the enquiry is as to his growth, and the relative proportion of the various parts of the body at different ages until his complete maturity. The last work of M. Quetelet, entitled "*Anthropométrie, ou mesure des différentes facultés de l'homme,*" published about the end of 1870, comprizes the results of many years of observations, in which, by the assistance of scientific friends, artists, and medical men, he has succeeded in collecting sufficient

trustworthy facts to trace the law of growth in every portion of the human body at all periods of life.

The methods formerly employed to ascertain the true proportions which constitute the typical man, were not satisfactory. Artists, and sculptors especially, whilst endeavouring to obtain models of perfection for the object they had in view, found it necessary to study nature with great care, and have left various observations on the proportions of the human body at different periods of life and in both sexes. But it does not in general appear clear how the measures were taken, nor whether the models chosen would fairly represent the form and proportions of man in general. If some typical form of the human race could be supposed to exist, the relative proportions of which were so fixed that any deviations from it in excess or defect could only arise from accidental causes, the observations recorded as to the measurements of any part of the body may be divided into groups, and according to the theory of probability the specific number which ought to be found in each group may be predicted beforehand with a very near approach to accuracy. The greater the number of observations, the more certainly will the observed number in each group agree with the number calculated by the theory. If the measures are divided into equal spaces, the group which approaches nearest to the mean will be the most numerous; and the other groups will be found to contain numbers as they differ from the mean in excess or defect in exact proportion to the coefficients of the terms of the binomial theorem.

This remarkable law, called for shortness the Binomial law, applies not only to the measurement of the height and growth of man, but to his weight, his strength, all his physical, moral, and intellectual actions, so far as they can be expressed in numbers,—to animals, to plants,—in fact, to all classes of facts capable of being classed by groups in numbers, and containing some typical form or character from which there are only what may be called accidental deviations.

In accordance with this law, when the extremes are known, dwarfs and giants are not casual monstrosities. If we obtain from a sufficient number of observations the height of certain groups of the population in any country, and out of these observations the giants and dwarfs had been purposely excluded, we should not only be able to state the numbers which had been omitted, but to assign to them their relative statures as compared with the rest of the people.

M. Quetelet devotes the second part of his work to a very interesting enquiry as to the measurement of man, as estimated by the proportions which artists in different countries and in ancient or modern times have given to the human form, male and female.

The work of Audran, published in Paris in 1683, entitled "*Les proportions du corps humain mesurées sur les plus belles statues de l'antiquité*," contains the proportions of Laocoon and his sons, of Pœtus, of the Egyptian Termes, of Antinous, of the Greek statue of Peace, of the Pythian Apollo, an antique fragment, and of two female figures. M. Quetelet has compared the proportions with those of the mean proportions obtained by the measurement of adult man in Belgium. The coincidences are striking, and the differences not greater than would be caused by errors of observation. The hand and the foot are slightly smaller amongst the Greeks than the Belgian model; but both differ from the common rule of the schools that the head and the foot are of equal length; for the Belgian model gives the former ·135 and the latter ·154 of the total height, and the Greek statues ·130 and ·149 respectively. All the measures tend to establish the fact that the proportions of the human form of the present day are almost identical with those deduced by observation from the most regular statues of Grecian art.

The proportions amongst the Romans are deduced from the rules laid down by Vitruvius, many of which have been since adopted by the moderns without naming the authority from which they proceed.

Pliny, after citing examples of extrêmes of stature, lays down the rule that the height of a man is the same as the distance between the tips of the middle fingers when the arms are stretched out in a strait line; and that at three years of age man will have reached half the height which he will eventually attain.

Amongst the Italian artists, Leone Battista Alberti, who was born in 1398, and lived 77 years, ascertained the proportions of the human body by selecting a great number of models reputed of fine and regular shape; and obtained a mean for the measurement of certain parts by rejecting what were in excess or defect, thus constructing as it were a type of the human form in theoretical proportions.

Various other Italian writers and artists of the highest celebrity have left their rules for the true proportions of man, but none appear to have taken the same simple and more natural method of Alberti, whereby the mean proportions from a considerable number

of measurements of each part are taken to represent the type of the human form. The German artists have in Albert Durer (born 1471, died 1528) the credit of possessing one who, amidst his celebrated works on mathematics and the arts, paid special attention to the present subject. Taking the height of man as the unit, he measured off the proportions of the various parts in length or diameter, in fractions of his unit. The female he makes $\frac{1}{18}$th part shorter than the man. M. Quetelet, by a great number of measurements, obtained 1·684 metres for the height of man, and 1·579 of the female, or $\frac{1}{16}$th part less. Albert Durer's measures of an infant about one year old, in which artists frequently fail as to the correct proportions, seem to have been carefully obtained from nature.

Schadow gave, from 1788 to 1848, frequent memoirs on the subject; but his principal work was "Polyclete, or theory of the measure of man according to sex and age." He claims the credit of having been the first to show the difference in the sexes of the head and the face, and the profile in maturity and old age, the measure of the infant just born, the proportions at various stages of growth in males, &c.

In none of the authors alluded to in these brief historical notices do we find the full application of those scientific principles on which M. Quetelet has himself established the harmony of proportions which constitute the type of man. The law of the growth of man is expressed by the binomial law of Newton, which, simple and general as it is, applies equally to his stature, his weight, and his physical and other powers. In the 40 years which have elapsed since M. Quetelet first pointed out its adaptation to the researches as to man, many facts have been added to strengthen his conclusions.

A remarkable confirmation is afforded in a paper presented by Mr. E. B. Elliott to the International Statistical Congress at Berlin in 1865, showing the measurement of the height of 25,878 volunteers in the United States, taken during the civil war. The intervals of height are taken as every 25 millimetres. The mean height is 1·75 metre, at the interval comprizing which the largest number of the men are found, 157 out of every 1000, when the calculated number would give 153 out of every 1000. At every other interval the calculated numbers correspond very closely to the observed numbers, the total deviations in 18 intervals being only -28 and $+28$ in every 1000 cases.

It is the result of observations that the extremes in the height

of men, if the human race were all measured, would be about 2·60 metres in one direction and 0·70 metre in the other. Between these limits, if all were grouped at intervals of 1 decimetre, the numbers at each group would be found not to be arbitrary or uncertain, but in a most remarkable manner to range themselves at equidistance from the mean, with a regularity and precision which may be calculated beforehand with the greatest accuracy from a few terms.

Two laws relating to the growth of man may be represented by two curves.

1. The ordinate, representing by its length men of mean height at each age, in passing along what may be called the line of life, will be found to touch in its upper point a descending hyperbolic curve, which may be called "the curve of mean height."
2. At any point in this curved line may be seen, on the right and left, the number of men of the same age, taller or shorter than those of the mean height, distributed with the greatest regularity in a plane, perpendicular to the line of mean height, and ranged according to a second line, which may be named the binomial line.

The two extremes of the last line would be the giants and the dwarfs at each age.

The expression for the curve representing by its abscissa and ordinate the individual height and the number of individuals of a given age, is in its most simple analytical form,

$$(a+b)^m = a^m + \frac{m}{1}\cdot a^{m-1}.b^1 + \frac{m(m-1)}{1.2}a^{m-2}b^2 + \dots\dots$$
$$+\frac{m(m-1)(m-2)\dots\dots(m-n+1)}{1.2.3\dots\dots n}a^{m-n}.b^n;$$

and when $a=b$,

$$2^m = 1 + \frac{m}{1} + \frac{m(m-1)}{1.2} + \frac{m(m-1)(m-2)}{1.2.3}\dots\dots$$

The middle term or mean would be

$$T = \frac{m(m-1)\dots\dots\left(\frac{m}{2}+1\right)}{1.2\dots\dots\frac{m}{2}}.$$

The simplicity of this formula rests upon the fact that every observed phenomenon must depend upon causes which are favourable or unfavourable to its happening. In one case these

causes may be equal in number, and in another the favourable causes may be more or less numerous than the contrary. In the former case a little table, showing in numbers the coefficient of each term of the binomial theorem for the number of terms or intervals required, with the number of chances possible, or 2^m, will show the number at each interval, and the least number of observations which ought to be collected to allow of all possible combinations of the event. This table may be conveniently reduced to a form in which the total chances or 2^m should be represented by 1000.

TABLE A.—*The Binomial Coefficients proportional to the total of* 1000 *as the Sum of the Terms.*

Number of terms. $m+1$	m																	Total Sum of the Actual Numbers $=2^m$.
2	1	500	500	...	...	...	...	...	...	...	...	...	...	...	...	...	...	2
3	2	250	500	250	...	...	...	...	...	...	...	...	...	...	...	...	...	4
4	3	125	375	375	125	...	...	...	...	...	...	...	...	...	...	...	...	8
5	4	62	250	375	250	62	...	...	...	...	...	...	...	...	...	...	...	16
6	5	31	156	313	313	156	31	...	...	...	...	...	...	...	...	...	...	32
7	6	16	94	234	312	234	94	16	...	...	...	...	...	...	...	...	...	64
8	7	8	55	164	273	273	164	55	8	...	...	...	...	...	...	...	...	128
9	8	4	31	109	219	274	219	109	31	4	...	...	...	...	...	...	...	256
10	9	2	18	70	164	246	246	164	70	18	2	...	...	...	...	...	...	512
11	10	1	10	44	117	205	246	205	117	44	10	1	...	...	...	...	...	1,024
12	11	...	5	27	80	161	226	226	161	80	27	5	...	...	...	...	...	2,048
13	12	...	3	16	54	121	193	226	193	121	54	16	3	...	...	...	...	4,096
14	13	...	1	9	35	87	157	210	210	157	87	35	9	1	...	...	...	8,192
15	14	...	1	6	22	61	122	183	210	183	122	61	22	6	1	...	...	16,384
16	15	...	1	3	14	42	91	153	196	196	91	42	14	3	1	...	...	32,768
17	16	...	...	2	8	28	67	122	175	196	175	122	67	28	8	2	...	65,536
18	17	...	...	1	6	18	47	95	148	185	185	148	95	47	18	6	1	131,072
19	18	...	...	...	3	11	33	71	121	167	185	167	121	71	33	11	3	262,144
20	19	...	...	...	2	8	22	52	96	144	176	176	144	96	52	22	8	524,288
21	20	...	...	...	1	4	15	37	74	120	160	176	160	120	74	37	15	1,048,576
22	21	...	...	...	1	3	10	26	55	97	140	168	168	140	97	55	26	2,097,152
23	22	...	...	...	...	2	6	18	41	76	119	154	168	154	119	76	41	4,194,304
24	23	...	...	...	...	1	4	12	29	59	97	137	161	161	137	97	59	8,388,608

In comparing facts of a similar kind in different countries, it would be advisable to take the same number of terms, and that they should be sufficiently numerous for accuracy of observations. Fifteen terms, for instance, would require at least 2^{15} or 32,768 observations in each set.

The following table gives the observed numbers and the numbers calculated by the above Law, of Men of different statures in America, France, Belgium, for 20 years of observation, and in Italy at 21 years of age. It shows in a very striking measure

how closely the observations agree with the number expected by theory.

Table B.—*Showing in the United States, France, Belgium, and Italy, the proportion of Conscripts of various heights, calculated to a total of* 1000.

Height (Metres).	United States, Volunteers.		United States, B. A. Gould.		France, M. D'Hargenvillers.		Belgium, A. Quetelet, 20 years' Observations.		Italy, at Age 21, M. Bodio.	
	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Number.	Calculated.
1·33								0·1	1	
1·36						0·5		0·3	3	
1·39						1·6		1·	13	
1·42	1	2				4·5		3·	56	1·5
1·45					286	11·	147·	7·	200	5·
1·48				1		24·		14·	799	22·
1·51	1	3	1	4		44·		28·	1762	47·
1·54	2	9	8	11		73·		53·	2932	80·
1·57	20	21	14	24	...	105·	110·	107·	4249	116·
1·60	48	42	49	45	116	132·	106·	136·	5535	150·
1·62	75	72	109	75	140	145·	162·	150·	5907	156·
1·65	117	107	93	109	144	140·	129·	150·	5535	150·
1·68	134	137	137	137	114	118·	138·	136·	4249	116·
1·70	157	153	148	150	88	87·	102·	107·	2932	80·
1·73	140	146	138	142	55	55·	48·	53·	1762	47·
1·76	121	121	112	117	32	32·	34·	28·	799	22·
1·79	80	86	99	84		16·	14·	14·	200	5·
1·81	57	53	45	52	25	7·	7·	7·	91	1·5
1·84	26	28	25	28		3·	2·	3·	35	
1·87	13	13	14	13		1·	0·6	1·	8	
1·90	5	5	7	5	...	0·3	0·3	0·3	1	
1·92	2	2	1	2	...	...	0·1	0·1		
1·94	1	0	...	1						
1·97										
2·00										
	1000	1000	1000	1000	1000	1000·	1000·	1000·	37069	1000·
Number observed	25878		761							

These measures are deduced from conscripts or men about to enter the military service. The tables differ only as to the "mean" height observed, which is nearly the same in France, Belgium, and Italy, but is higher by 6 or 8 centimetres in the United States, though this may be in part attributed to the fact that the volunteers in the latter were of a more mature age.

The same law applies to measurements of parts of the body. In the following table is shown the circumference of the chest of Scottish soldiers and of American soldiers (of the Potomac). The mean of the Scotchman is 40 inches, at which the maximum number was 199 out of every 1000 examined, and the mean of the

circumference of the American soldiers' chest, was 35 inches, at which the maximum number of 190 out of 1000 were observed; and both agree very closely with the binomial numbers for 16 terms.

TABLE C.—*Showing in English and metrical measures the circumference of the chest of Scottish and American Soldiers.*

Measure.		Scottish Soldiers.			American Soldiers (of the Potomac).		
English Inches.	Metre.	Number of Men.	Observed.	Calculated.	Number of Men.	Observed.	Calculated.
28	·711	...	...	...	2	1	1
29	·736	...	...	...	4	3	3
30	·762	...	...	...	17	11	11
31	·787	...	...	...	55	36	32
32	·813	...	...	...	102	67	69
33	·838	3	1	1	180	119	121
34	·864	18	3	3	242	160	170
35	·889	81	14	11	310	204	190
36	·914	185	32	32	251	166	169
37	·940	420	73	73	181	119	120
38	·965	749	130	133	103	68	68
39	·991	1073	187	184	42	28	31
40	1·016	1079	188	199	19	13	11
41	1·041	934	163	167	6	4	3
42	1·067	658	115	110	2	1	1
43	1·092	370	65	56			
44	1·118	92	16	22			
45	1·143	50	9	7			
46	1·169	21	4	2			
47	1·195	4	1				
48	1·220	1					
		5738	1000	1000	1516	1000	1000

In considering the mean stature of man the proportions of the two extremes are worthy of attention. One remark may be made, that, as a general rule, in the large man the head is small relatively to the rest of the body, and in the small man the reverse. In measuring a Neapolitan giant of 2·15 metres in height, against the *élite* of the Belgian army (10 men of the Regiment of Guides), he exceeded the latter by nearly one quarter. The size of the foot and the hand and the length of the limbs were nearly in the same proportion, but the length of the head was only greater by $\frac{9}{100}$ parts.

In computing from the observations on French conscripts, the proportions at the greatest height, it appeared that we should only expect to find one in ten millions of men to exceed 2·027 metres in height (6 feet 8 inches).

M. Quetelet cites various examples of dwarfs—the well known "General Tom Thumb," "Admiral van Tromp," and a dwarf born

in 1838 in the village of Kerkum near Tirlemont, in Belgium, compared with the dimensions of an infant 13 to 15 months old, the three former being more than 11 years old. The heights were respectively ·710, ·728, ·786 metre, the infant being ·710. Most of the other measurements are in near proportion to those of the infant. Other dwarfs have been quoted in natural history.

Bébé, the dwarf of King Stanislaus	·893	metre.
A Polish gentleman, Joseph Burwilaski (1760)	·758	„
A dwarf at Bristol (1751)	·787	„
A peasant in Frizeland (1751)...	·684	„
A dwarf in Norfolk	·965	„
And one quoted by Buffon (from Birch) as only	·433	„

The last one is the smallest mentioned, and the measure seems doubtful.

On the other hand, the giant in the Gards du Corps of Frederick the Great, King of Prussia, 8½ Swedish feet or 2·523 metres, had the greatest height authenticated.

The giant at Paris (1755)	2·184	metres.
The giant at Thorseby, England ...	2·261	„
The porter of the Duke of Wurtemberg	2·354	„
Three other English giants	2·286	„
„	2·311	„
„	2·337	„
The giant Cujanus, in Sweden	2·375	„
A Swedish peasant	2·375	„
One of the Guard of the Duke of Brunswick, Hanover	2·406	„
The giant Gills, of Trent, in the Tyrol	2·424	„

Taking the mean height of man in France 1·617 metre, the giant of the King of Prussia would exceed it by ·906m., and General Tom Thumb would be less by ·907, which would be the extremes, for the other one cited seems almost beyond the limits of credibility.

	Highest Limit.	Diff.	Mean.	Diff.	Lowest Limit.
For the height of children just born Buffon gives the measurement...	·596	·109	·487	·108	·379
At the Hospice de Maternité, in Bruxelles, on 50 males	·532	·032	·500	·062	·438
Ditto, on 50 females	·555	·059	·496	·058	·438
At Moscow, on 44 new-born children, both sexes	·569	·068	·501	·095	·406

From the above extremes at all ages of maturity, and from the height of the new-born infant, the law of the growth of man may be traced at every successive age to its full development. In an elaborate series of tables M. Quetelet gives at every age from birth the mean of the height and of the absolute measurements of all

parts of the body; and in order to show at what ages any rapid increase or check in growth of any part may occur, there are also given the proportions of each part, taking the height as the unit.

The table is too long to produce in full; but its leading features may be seen by the following summary at intervals of five years of age from birth to maturity, of the height, the length of some of the members, and the circumference of the head and body.

TABLE D. *Growth of Man at different Ages.*

	Actual Measures (in metres and millimetres). Ages.														Proportion at maturity to birth.	
	0.		5.		10.		15.		20.		25.		30.			
	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.
Height	·500	·494	·987	·974	1·273	1·249	1·513	1·488	1·669	1·574	1·682	1·578	1·686	1·580	3·37	3·20
Height of the head (vertex to the chin)	·111	·111	·192	·188	·205	·201	·215	·213	·227	·220	·228	·221	·228	·221	2·06	1·99
Top of the head to the clavicles .	·140	·139	·207	·204	·237	·231	·266	·259	·285	·270	·287	·270	·288	·270	2·06	1·94
Clavicles to bifurcation (torso) .	·200	..	·376	..	·451	..	·524	..	·585	..	·587	..	·590	..	2·95	..
Bifurcation to the ground . .	·160	·158	·404	·396	·584	·570	·723	·698	·802	·737	·806	·739	·806	·739	5·03	4·67
Length of the arm. . . .	·206	·204	·422	·412	·556	·532	·675	·644	·758	·694	·764	·697	·766	·698	3·71	3·44
„ „ hand . . .	·061	·060	·113	·112	·143	·137	·171	·167	·188	·176	·189	·177	·190	·177	3·11	2·95
„ „ foot. . . .	·075	·074	·157	·152	·204	·197	·245	·230	·264	·236	·264	·236	·264	·237	3·52	3·20
Circumference of the head (orbits)	·335	·335	·503	·500	·527	·517	·547	·523	·564	·535	·564	·537	·564	·538	1·69	1·61
„ by the shoulders .	·321	·317	·564	·554	·685	·654	·788	·768	·924	·843	·936	·855	·937	·856	2·93	2·70
„ by the waist . .	·281	·278	·527	·503	·585	·540	·670	·613	·741	·661	·746	·670	·748	·679	2·66	2·44

	Relative Measures (taking the Height at each Age for Unity). Ages.														Proportion at maturity to birth.	
	0.		5.		10.		15.		20.		25.		30.			
	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.	M.	F.
Height of the head (vertex to the chin)	·231	·233	·195	·197	·161	·163	·145	·141	·138	·139	·137	·140	·136	·140	·59	·60
Top of the head to the clavicles .	·280	·280	·210	·210	·187	·185	·176	·175	·171	·171	·171	·171	·171	·171	·61	·61
Clavicles to bifurcation (torso) .	·400	..	·381	..	·353	..	·346	..	·349	..	·349	..	·350	..	·88	..
Bifurcation to the ground . .	·320	·320	·409	·407	·460	·456	·478	·469	·480	·469	·480	·468	·479	·467	1·50	1·48
Length of the arm. . . .	·412	·412	·427	·423	·437	·426	·446	·433	·454	·442	·455	·442	·455	·442	1·10	1·07
„ „ hand . . .	·122	·122	·115	·115	·113	·110	·113	·111	·113	·113	·113	·113	·113	·113	·93	·93
„ „ foot	·150	·150	·159	·157	·161	·158	·162	·155	·158	·150	·157	·149	·157	·149	1·05	1·00
Circumference of the head (orbits)	·670	·676	·510	·512	·415	·414	·361	·351	·338	·340	·336	·340	·335	·340	·50	·50
„ by the shoulders .	·642	·642	·571	·568	·539	·523	·521	·516	·553	·537	·558	·540	·558	·542	·87	·84
„ by the waist . .	·562	·562	·533	·516	·460	·433	·443	·412	·444	·421	·444	·425	·444	·430	·79	·77

Computations from these tables indicate that the growth of man virtually terminates at 30 years of age, though the difference between 23 or 25 and 30 is very small indeed; that up to 4 years of age the female is shorter than the male in the proportion of 1 to ·988, and that at maturity the difference is greater, being 1 to ·938; that after 50 years of age there is a diminution in height in both sexes, the total till old age being, in Belgium, from 1·686 to

1·610m. for males, or 76 millimetres, and from 1·580m. to 1·510m., or 70 millimetres for females; that at birth the infant has about ⅓rd of the future height, about 7 years of age ⅔rds, and about 10 years of age ⅘ths.

Another table enables us by means of the limits and the mean of height, taken from observation in each two consecutive ages up to 20, then for the five years up to 25, to deduce the number that should be found living in Belgium for every age and every interval of height between the given limits. It is much to be desired that other countries would supply the facts and accumulate materials for extending the enquiry. But they should be so collected as to eliminate the causes which influence the growth of man,—as for instance, locality, whether town or country, residence in mountains or plains, moderate or excessive labour, agricultural or manufacturing employments, the quality and quantity of food obtainable, the seasons, &c. In this way, the application of the true scientific method proposed for obtaining the mean result of the causes at work, would indicate, by the aberrations at any point in the curve, where we must look for some disturbing cause.

The investigations as to the weight, the physical force, pulsations, and inspirations, depending on the age and growth of man, have already made some progress; but much more may be expected when more facts are collected by scientific men using the true theory for their method of search. But the extension of this method of enquiry into the actions which appear to be governed only by the free will of man, will reveal still more curious and striking results. In these cases, the difficulty in the first instance is to define precisely the event or action by which his free will is exhibited. There is so much subtleness in the enquiry, so many causes may be at work to produce the result, that it would seem at first sight almost impossible to come to any definite conclusions. To ascertain the law of mortality would seem comparatively easy, since we deal with a definite fact—the death of the individual. But in the research for the causes of mortality, for instance, we have first to settle the nomenclature of diseases, and then be subject to the skill and judgment of the physician in reading the disease aright. This indefiniteness must be greater still in judging of the moral or intellectual actions of man. But time and thought will enable us to lay down some general principles for this new branch of statistics, and in the meantime a commencement may be made by taking one class of facts which is common to all civilized nations. Whatever diversity of laws may be at work to encourage or restrain marriage,

the actual celebration of it in any form constitutes a definite fact, which enables us, by a sufficient collection of the statistics, when compared together, to ascertain whether they modify to any and to what extent this expression of the free will of man. Several tables have now been given in the Government Reports of different countries. It is shown that in Belgium, in 5 consecutive Quinquennial periods of years from 1841 to 1865, the proportion of men of certain ages marrying with women of certain ages, has varied very little indeed from the mean number in each period at whatever age we examine the table, the number of marriages in each year being all compared with a total of 10,000 in each year.

A still more striking coincidence will be found in the Registrar-General's Reports in this country, in which the details are given for each group of Quinquennial ages, and which I have grouped together so as to show the proportion in 10,000 marriages of the marriages for every 5 consecutive ages, and on the averages of two periods of 3 years each—1846 to 1848 and 1851 to 1853. (See *Assurance Magazine and Journal of the Institute of Actuaries*, vol. vii, p. 188.)

The total numbers observed are very large—81,964 in the first period, 231,797 in the second. The results are most remarkable, indicating that, notwithstanding all the changes in the social condition of the country and the apparent caprices of the individual, there is a natural law of marriage depending upon age, which (notwithstanding the free agency of man) acts with a regularity which imperceptibly confines his actions within limits which vary with his age.

Another class of facts which are sufficiently definite in their character, is that which comprizes the tendency to crime, shown by the number of accused or condemned persons in a given period of time out of a given population. These clearly indicate that the principal subdivision must be under ages. It will then be seen that certain classes of crime are more prevalent at one age than another, and the comparisons from year to year ought to show whether the influence of education or legislation, or any other causes are at work to improve or deteriorate the condition of society. A large mass of materials has been collected in many countries, but it is to be feared they have not yet been subjected to scientific analysis. M. Quetelet has shown in a series of valuable tables how these principles ought to be applied, and what important and interesting deductions may be drawn from them.

On Errors in Tables of Logarithms of Numbers. By J. W. L. Glaisher, B.A., *Fellow of Trinity College, Cambridge.*

THE following results compiled from my paper "On the Progress to Accuracy of Logarithmic Tables," read before the Royal Astronomical Society, on 13th December 1872, will probably be found interesting to the readers of the *Journal of the Institute of Actuaries,* more especially in connection with the recent correspondence in reference to Mr. Sang's logarithmic tables.

I have made a careful examination of Vlacq's *Arithmetica Logarithmica* of 1628 (from which every subsequent table, except the last half of Mr. Sang's, has been copied directly or indirectly), by means of all the published errata-lists, and have made a complete list of the errors contained in it which can affect a seven-figure table. They are 171 in number; but of these 48 occur in the first 10,000 logarithms, which are omitted in subsequent tables, the usual range of which is from 10,000 to 100,000. There are thus 123 errors left in Vlacq, which have been gradually found out and corrected in the 250 years that have elapsed; and the matter may be viewed in the following light.

A table is published containing a certain number of errors (miscalculations, misprints, &c.); it is greatly and extensively used in all the sciences, and repeatedly reprinted during two centuries and a half, though never re-calculated. It is then a matter of a good deal of interest to watch the progress of accuracy with the lapse of time, and see how long the struggle lasted till accuracy gained the victory: a triumph not quite complete even yet. The following list was formed by comparing the 123 errors in Vlacq (which, with their corrections, are given in the paper mentioned above), with the works named, and noting the number that remained uncorrected. In all cases the logarithms compared were those belonging to the numbers from 10,000 to 100,000 only (one or two tables begin at 12,000, but the intermediate 2000 contain no errors that were ever reproduced, except in the tables immediately following Vlacq), and the tables are to seven places, unless otherwise stated. The list of tables is not given as perfect; in fact no attempt at completeness has been made, as only works that I could easily put my hand upon are included. Roe's work (1633), the first seven-place table, is omitted, as the seventh figure is not increased when the succeeding figures exceed 500, so that the comparison is difficult.

Vlacq, Gouda, 1628, and London, 1631 (ten-place tables)	123
John Newton, London, 1658 (eight-place table)	98
Sherwin, London, 1706	65
„ „ 1726	58
„ „ 1741	20
Gardiner, London, 1742	19
Gardiner (reprint by Pezenas), Avignon, 1770	17
Sherwin, London, 1771	20
Hutton, London, 1785	12
Taylor, London, 1792	6
Callet, Paris, 1793	11
Vega, Leipzig, 1794 (*Thesaurus*, &c., ten-place table)	23
Hutton, London, 1794	10
Vega (*Tabulæ, &c.*), Leipzig, 1797	5
Hutton, London, 1801	5
Hutton, London, 1822	5
Babbage, London, 1827	1
Hassler, New York, 1830	2
Babbage, London, 1831	1
Hülsse (small edition of Vega), Leipzig, 1840	1
Shortrede, Edinburgh, 1844	7
Babbage, London, 1841	1
Bell (Chambers's Educational Course), Edinburgh, 1847	6
Shortrede, Edinburgh, 1849	6
Anonymous (Chambers's Educational Course), Edinburgh, 1853	6
Hutton (edited by Gregory), London, 1855	4
Callet, Paris, 1855	2
Bremiker (edition of Vega; English edition by Fischer) Berlin, 1857	0
Schrön, Braunschweig, 1860	0
Callet (revue par Dupuis), Paris, 1862	0
Bruhns, Leipzig, 1870	0
Sang, Edinburgh, 1871	2

Two points with respect to the above list must be particularly noticed; (1) that the numbers given refer to the tables *as printed*, all errata lists or corrections contained in the preface, &c., being ignored. This rule was rendered necessary by the fact that errors in tables were often discovered before the whole impression was made up, so that copies of the same edition, and with the same date, have often different errata lists prefixed; and (2) that the list shows only the number of Vlacq's errors that are reproduced in each of the works named, and does not convey any information with regard to the total number of errors in any of these works; it represents, so to speak, the editor's knowledge and research, but not his (or the printer's) care in reading the proofs. Thus, besides the numbers given above, there might, as far as the list is

concerned, be any number of fresh errors introduced by careless copying and revising. This being premised, the list shows the progress to accuracy of tables of logarithms of numbers. Of Vlacq's 123 errors he corrected 34 in an errata list, prefixed to his work, and it has taken more than two centuries to eliminate the remaining 89; and even now only the best tables are quite free from them. The great number of errors assigned to Vega (1794) is very curious, as he is generally regarded, and justly, as the great purifier of tables of logarithms of numbers. The discrepancy is explained by the fact that I have ignored the errata list he published himself; he revised his table after it was printed, and also offered a ducat for every error found, so that the errata lists that were appended to the last copies of the impression of 1794 are very complete, and include every one of the 23 errors noted above. Thus Vega, in 1794, was in possession of every error that could affect a seven-place table; though from the above list it appears that 1857 is the date of the first table which has come under my notice that is free from hereditary error. The errors in log 52943 and log 38962 were the last to die out; the former always occurs in every work that contains an error at all (except Vega, 1797), and the latter in every table that contains at least two errors. One remarkable fact that follows from the list is, that seven-figure tables are not in any way indebted for their accuracy to the comparisons with the great French M.S. Tables (*Tables du Cadastre*), as all the errors were known by the very year in which the calculation of the latter was ordered. The list shows the effect of a complete free trade in tables; as, had there been any *permanent* body (such as the Old Board of Longitude, or the Royal Society) to receive and publish errata, the tables would have become accurate long ago. Tables *will* contain errors, and if there is no permanent body that will receive them as found out, they will continue to be reproduced. The British Association Committee on Mathematical Tables will endeavour to collect errors in existing tables; but the Committee cannot, of course, be permanent.

On Mathematical Statistics and its application to Political Economy and Insurance. *By* Dr. THEODOR WITTSTEIN, *Actuary of the Hanover Life Insurance Company.* *Translated by* T. B. SPRAGUE, M.A.

(Continued from page 189.)

§ 6.

Besides the probable error above considered, there is another quantity of some importance in practice, which may be called the Risk of a departure from the most probable number of survivers. It is obtained by adding together the products of all the possible positive deviations in the number of the survivers from its most probable value, each multiplied into the probability of its occurrence; or by adding together the products of all the possible negative deviations in the number of the survivers from its most probable value, each multiplied into the probability of its occurrence; in this case, however, taken without sign. For if we consider the deviations from the most probable number of survivers, taken without regard to sign, as represented by proportional sums of money, then the quantity just defined evidently expresses either the mathematical expectation of gain of a person who is to receive those sums, or the mathematical expectation of loss of a person who is to pay them. The latter of these two ideas more generally occurs in practice. If we denote this Risk by R, we have from (6)

$$R = \frac{h}{\sqrt{\pi}}\int_0^\infty e^{-h^2k^2}kdk = -\frac{h}{\sqrt{\pi}}\int_{-\infty}^0 \epsilon^{-h^2k^2}kdk = \frac{1}{2h\sqrt{\pi}}$$

whence by (4)

$$R = \frac{1}{\sqrt{2\pi}}\sqrt{lp(1-p)} = \cdot 3989\sqrt{lp(1-p)} \quad . \quad . \quad . \quad (10)$$

This formula differs only in its numerical coefficient from that for the probable error, (8); and the observations in §§ 4 and 5 can therefore be easily extended to this quantity.

For instance, we have according to Brune's table, out of 7943 males living at the age 40, 7847 living at the end of a year, with a Risk $=3\cdot 89$; or if an Office, as usually happens, insures a given sum for every death among the 7943 males, it must charge, in addition to the most probable amount of the claims, 96, a sum of 3·89 for the risk, that is, nearly 4 per-cent additional.

This has an important bearing on the calculation of the premiums of life insurance companies. Hitherto, every calculation of life insurance premiums has been based only upon the most

probable number of survivers; and a loading of arbitrary magnitude has been added to the premiums to cover the Risk. If, instead of this, in calculating any premium we were to add to the most probable value of the premium found in the usual way, the corresponding "risk" found by means of the formula (10), we should then have a perfectly intelligible determination of the magnitude of the loading necessary to cover the risk arising from fluctuations in the mortality. Similar remarks apply as regards the value of the policy and the Guarantee Fund that is necessary to cover the risk in question. The details of these calculations I reserve for another opportunity.

Chapter 2.

The probability of living a year is to be deduced from observation.

§ 7.

The probability of living a year is never known *à priori,* but must be deduced from facts furnished by experience. But since this can never be done with certainty, but only with a greater or less degree of probability, this same degree of probability must also attach to the final result; and the above conclusions consequently require to be modified.

Suppose that observation of L persons of the age x has shown us L′ of them living at the end of a year. Let y be the unknown probability that a person of the age x will be alive at the end of a year. If y were known, the probability of the above observed fact would be, as in (1),

$$\frac{\lfloor L}{\lfloor L' \lfloor L-L'} y^{L'}(1-y)^{L-L'} \quad . \quad . \quad . \quad . \quad (11)$$

Consequently, the probability, Ω, of the hypothesis y, is

$$\Omega = \frac{y^{L'}(1-y)^{L-L'}}{\int_0^1 y^{L'}(1-y)^{L-L'}dy}$$

that is to say, $$\Omega = \frac{\lfloor L+1}{\lfloor L' \lfloor L-L'} y^{L'}(1-y)^{L-L'} \quad . \quad . \quad (12)$$

where y can have all values from 0 to 1.

The value of y for which Ω is a maximum is

$$y = \frac{L'}{L}.$$

If we denote by y_0 this *most probable value* of the probability y, we have

$$y_0=\frac{L'}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

This value is what is commonly, but erroneously, regarded in practice as the *true value* of the probability sought.

§ 8.

If observations of the above kind have been made for various years, and we desire to deduce a single result from them, then the above investigation must be modified as follows:

Suppose we have observed that

In the 1st year,	out of l	persons of the age x,	l'	are alive at the end	of the year,	
" 2nd "	m	"	" m'	"	"	
" 3rd "	n	"	" n'	"	"	

and so on.

Let y be the unknown probability that a person of the age x will be alive at the end of a year. Assuming the hypothesis y, the probabilities of the facts observed in the several years are, by (11),

$$\frac{\underline{|l}}{\underline{|l'}\,\underline{|l-l'}}\,y^{l'}(1-y)^{l-l'},\quad \frac{\underline{|m}}{\underline{|m'}\,\underline{|m-m'}}\,y^{m'}(1-y)^{m-m'},$$
$$\frac{\underline{|n}}{\underline{|n'}\,\underline{|n-n'}}\,y^{n'}(1-y)^{n-n'},\ \ldots.$$

Consequently, the probability of the concurrence of all these observed facts is equal to the product

$$\left.\begin{array}{l}\dfrac{\underline{|l}}{\underline{|l'}\,\underline{|l-l'}}\cdot\dfrac{\underline{|m}}{\underline{|m'}\,\underline{|m-m'}}\cdot\dfrac{\underline{|n}}{\underline{|n'}\,\underline{|n-n'}}\cdots\ldots \\ \qquad\times\, y^{l'+m'+n'+\ldots\ldots}(1-y)^{l+m+n+\ldots\ldots-l'-m'-n'-\ldots\ldots}\end{array}\right\}\ (14)$$

Hence, putting
$$\left.\begin{array}{l} l+m+n+\ \ldots\ldots=L \\ l'+m'+n'+\ \ldots\ldots=L'\end{array}\right\}\quad . \quad . \quad . \quad . \qquad (15)$$

we get for the probability of the hypothesis y, the same formula as before, (12); and for the most probable value of y, the formula (13). The result of the investigation therefore is, that we must treat observations made in different successive years as if they were made at the same time upon as many coexistent bodies of persons, and combine these into a single body;—a course which in itself appears reasonable.

Furthermore, this investigation plainly assumes that there is no

reason to suppose that the value of the probability y is different in different years.

§ 9.

If persons have come under observation or gone out of observation (entered or exited) during the year, so as to be under observation for part of a year only, these can be regarded as incomplete observations, and be taken into account as follows:—

Suppose we find from observation

	A persons	living at the beginning of the year,
	B ,,	entering in the course of the year,
	C ,,	exiting ,, ,,
	D ,,	dying ,, ,,
and lastly,	A′ ,,	living at the end of the year,

so that we have $A' = A + B - C - D$;

all these persons being supposed to be of the same age x at the beginning of the year. Assume now that the entries and exits are distributed uniformly over the year, and let q be the probability that a person of the age x will die in the course of the year.

If a fraction, θ, of the year has elapsed, the probability of dying in the remaining part of the year may be put proportional to that part,* that is, $= q(1-\theta)$. Now, out of A persons alive at the beginning of the year, the most probable number of persons dying in the course of the year is Aq.

Furthermore, out of B persons entering in succession, we have as the most probable number of persons dying after entry

$$\int_0^1 Bq(1-\theta)d\theta = \frac{B}{2}q$$

that is to say, just as many as would die out of $\frac{B}{2}$ living at the beginning of the year.

Lastly, out of C persons exiting in succession, the most probable number of persons dying after exit is

$$\int_0^1 Cq(1-\theta)d\theta = \frac{C}{2}q$$

that is to say, just as many as would die out of $\frac{C}{2}$ living at the beginning of the year.

Consequently, $$Aq + \frac{B}{2}q - \frac{C}{2}q \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

* See my paper, "*Die Mortalität in Gesellschaften mit successiv eintretenden und ausscheidenden Mitgliedern,*" in Grunert's Archiv, 39. Theil. 1. Heft.

is the most probable number of persons dying in the course of the year. But with the notation of § 7, this same value is expressed by Lq; so that we must put

$$\left.\begin{array}{ll} & L=A+\dfrac{B-C}{2} \\ \text{Also, since } L'=L-D & \\ \text{we have} & L'=A'-\dfrac{B-C}{2} \end{array}\right\} \quad \dots\dots \quad (17)$$

and the problem is thus reduced to that of § 7.

Observation. These formulas (17) must evidently not be applied if the supposition we have made above does not hold good, and the entries and exits, instead of being uniformly distributed over the year, fall at fixed times in the year. In such a case, it is clear from the foregoing, that the numbers of persons entering or exiting must be multiplied by the fraction of a year for which they were under observation, and be added to or subtracted from the number at the beginning of the year, in order to get the proper value of L.

Thus, for example, the exits in the Berlin Widows' Fund, which furnished the materials for the construction of Brune's table of mortality, took place only at the middle and the end of the year. In that case, therefore, dividing the exits of each year equally between the middle and the end of the year, we must put $L=A-\frac{1}{4}C$, and consequently $L'=A'+\frac{3}{4}C$.

§ 10.

Going back now to § 7, if we put $y=y_0+u=\dfrac{L'}{L}+u$, where u may have all values from $-\dfrac{L'}{L}$ to $\dfrac{L-L'}{L}$, then by (12)

$$\Omega=\frac{\lfloor L+1}{\lfloor L' \lfloor L-L'}\left(\frac{L'}{L}+u\right)^{L'}\left(\frac{L-L'}{L}-u\right)^{L-L'}$$

$$=\frac{(L+1)\lfloor L}{\lfloor L' \lfloor L-L'}\left(\frac{L'}{L}\right)^{L'}\left(1+\frac{Lu}{L'}\right)^{L'}\left(\frac{L-L'}{L}\right)^{L-L'}\left(1-\frac{Lu}{L-L'}\right)^{L-L'}$$

whence, with the help of Stirling's formula, if L and L′ are large numbers, we get

$$\Omega=\sqrt{\frac{L^3}{2(L-L')L'\pi}}\cdot\left(1+\frac{Lu}{L'}\right)^{L'}\left(1-\frac{Lu}{L-L'}\right)^{L-L'} \quad \dots \quad (18)$$

as the approximate formula for the probability that the value of y will differ from its most probable value by the amount u.

If we consider only values of u which are small in comparison with unity, we can expand the hyperbolic logarithm of the last two factors into

$$Lu-\tfrac{1}{2}\frac{L^2u^2}{L'}+\ldots\ldots-Lu-\tfrac{1}{2}\frac{L^2u^2}{L-L'}-\ldots\ldots$$

or

$$-\frac{L^3u^2}{2L'(L-L')}+\ldots\ldots$$

Hence, putting for brevity,

$$\frac{L^3}{2L'(L-L')}=h^2 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

(18) becomes

$$\Omega=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2u^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

It will be noticed that we hence have, since the extreme values of which u admits are $-\infty$ and ∞,

$$\int_{-\infty}^{+\infty}\Omega du=1.$$

§ 11.

The probability that the unknown value of the probability y lies between the limits $\frac{L'}{L}+u$ and $\frac{L'}{L}-u$, becomes by (20)

$$=\frac{2h}{\sqrt{\pi}}\int_0^u \epsilon^{-h^2u^2}du \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Let this probability take the value $\frac{1}{2}$, and let s be the value of u for which this happens, then we get the formula for the *probable error* to be expected in the determination of the probability y,

$$s=\frac{\cdot 6745}{h\sqrt{2}}$$

$$=\cdot 6745\sqrt{\frac{L'(L-L')}{L^3}} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

or (see the end of § 7),

$$s=\cdot 6745\sqrt{\frac{y_0(1-y_0)}{L}} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Hence, as the number of observations increases, the probable error varies inversely as the square root of the number of lives observed. It can therefore, by assuming a sufficiently large number of observed lives, be reduced to any degree of smallness that we choose.

For instance, according to the experience of the Berlin Widows' Fund, out of 21,734 males of the age 40 under observation at the beginning of the year, 317 exited and 271 died in the course of a year. (See the *Allgemeine Versicherungszeitung,* 1847). According to § 9, *Observation,* this comes to the same thing as if out of 21,654·75 males there are 21,383·75 still alive at the end of a year. Hence we get for the most probable value of the probability of a male of 40 being alive at the end of a year

$$y_0 = \cdot 98749\,;$$

and for the probable error of this value

$$s = \cdot 00051\,;$$

or it is an even chance that the probability in question lies between ·98698 and ·98800. The value given by Brune's table,

$$\frac{7847}{7943} = \cdot 98792$$

is contained within this interval.

§ 12.

By means of (12) we find the probability that any other person of the same age x who may be observed will be alive at the end of a year

$$= \frac{\int_0^1 y^{L'+1}(1-y)^{L-L'}dy}{\int_0^1 y^{L'}(1-y)^{L-L'}dy} = \frac{L'+1}{L+2} \quad . \quad . \quad (24)$$

This is different from the value $y_0 = \frac{L'}{L}$, which, as we saw, makes the value of Ω in (12) a maximum; and the two can only be considered equal to each other in the case of large numbers.

§ 13.

Again, by means of (12) we find the probability that out of l other persons of the same age x, observed for a year, l' will be alive at the end of the year,

$$\mathrm{P} = \frac{\lfloor l}{\lfloor l' \lfloor l-l'} \cdot \frac{\int_0^1 y^{L'+l'}(1-y)^{L+l-L'-l'}dy}{\int_0^1 y^{L'}(1-y)^{L-L'}dy} \cdot$$

$$= \frac{\lfloor l}{\lfloor l' \lfloor l-l'} \cdot \frac{\lfloor L'+l' \lfloor L+l-L'-l'}{\lfloor L+l+1} \cdot \frac{\lfloor L+1}{\lfloor L' \lfloor L-L'} \quad . \quad . \quad (25)$$

Here l' can have any value from 0 to l. The value of l' for

which P is a maximum may be found as in § 1, and is given by the conditions

$$\frac{l-l'+1}{l'}\cdot\frac{L'+l'}{L+l-L'-l'+1}>1>\frac{l-l'}{l'+1}\cdot\frac{L'+l'+1}{L+l-L'-l'}$$

that is
$$\frac{l'+1}{l+1}>\frac{L'}{L}>\frac{l'}{l+1};$$

whence, if L, L′, and l are large numbers, and l_0 denotes the most probable number of survivers, we have

$$l_0=\frac{L'l}{L}\,.\quad\ldots\ldots\ldots\qquad(26)$$

§ 14.

Putting $l'=l_0+k=\frac{L'l}{L}+k$, the probability that the actual number of survivers will differ from the most probable number by k, is found by means of (25) to be

$$\frac{\underline{|l}}{\underline{\left|\frac{L'l}{L}+k\right.}\;\underline{\left|l-\frac{L'l}{L}-k\right.}}\cdot\frac{\underline{\left|L'+\frac{L'l}{L}+k\right.}\;\underline{\left|L+l-L'-\frac{L'l}{L}-k\right.}}{\underline{|L+l+1}}\cdot\frac{\underline{|L+1}}{\underline{|L'}\;\underline{|L-L'}}\quad\ldots\qquad(27)$$

Here k can have any value from $-\frac{L'l}{L}$ to $\frac{(L-L')l}{L}$.

If we suppose L, L′, and l to be large numbers, and proceed as in § 2, this formula can be transformed into

$$P=\frac{1}{\sqrt{2\pi}}\times\frac{l^{l+\frac{1}{2}}}{\left(\frac{L'l}{L}+k\right)^{\frac{L'l}{L}+k+\frac{1}{2}}\left\{\frac{(L-L')l}{L}-k\right\}^{\frac{(L-L')l}{L}-k+\frac{1}{2}}}$$
$$\times\frac{\left\{\frac{L'(L+l)}{L}+k\right\}^{\frac{L'(L+l)}{L}+k+\frac{1}{2}}\left\{\frac{(L-L')(L+l)}{L}-k\right\}^{\frac{(L-L')(L+l)}{L}-k+\frac{1}{2}}}{(L+l+1)(L+l)^{L+l+\frac{1}{2}}}$$
$$\times\frac{(L+1)L^{L+\frac{1}{2}}}{L'^{L'+\frac{1}{2}}(L-L')^{L-L'+\frac{1}{2}}}$$
$$=\sqrt{\frac{L^3}{2lL'(L-L')(L+l)\pi}}\times\sqrt{\frac{\left\{1+\frac{Lk}{L'(L+l)}\right\}\left\{1-\frac{Lk}{(L-L')(L+l)}\right\}}{\left(1+\frac{Lk}{L'l}\right)\left\{1-\frac{Lk}{(L-L')l}\right\}}}$$
$$\times\frac{\left\{1+\frac{Lk}{L'(L+l)}\right\}^{\frac{L'(L+l)}{L}+k}\left\{1-\frac{Lk}{(L-L')(L+l)}\right\}^{\frac{(L-L')(L+l)}{L}-k}}{\left(1+\frac{Lk}{L'l}\right)^{\frac{L'l}{L}+k}\left\{1-\frac{Lk}{(L-L')l}\right\}^{\frac{(L-L')l}{L}-k}}.$$

Extracting the square root, the second factor becomes

$$1+\frac{L^2(2L'-L)k}{2lL'(L-L')(L+l)}+\dots\,.$$

and the Naperian logarithm of the third factor is

$$-\frac{L^3k^2}{2lL'(L-L')(L+l)}+\dots\,.$$

Hence, putting $$\frac{L^3}{2lL'(L-L')(L+l)}=h^2 \quad . \quad . \quad . \quad . \quad (28)$$

the above formula becomes

$$P=\frac{h}{\sqrt{\pi}}\left\{1+\frac{(2L'-L)h^2k}{L}\right\}\epsilon^{-h^2k^2} \quad . \quad . \quad . \quad (29)$$

or, since we need only regard values of k which are small in comparison with L, L$'$, and l,

$$P=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence, again, since the extreme values which k can have extend from $-\infty$ to $+\infty$, we have

$$\int_{-\infty}^{+\infty}Pdk=1.$$

§ 15.

The probability that the number of survivers will lie between $\frac{L'l}{L}+k$ and $\frac{L'l}{L}-k$, is by (30),

$$=\frac{2h}{\sqrt{\pi}}\int_0^k\epsilon^{-h^2k^2}dk \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

If σ' is the value of k for which this probability is equal to $\frac{1}{2}$, we get the formula for the *probable error* to be expected in the number of survivers

$$\sigma'=\frac{\cdot6745}{h\sqrt{2}}$$

$$=\cdot6745\sqrt{\frac{lL'(L-L')(L+l)}{L^3}} \quad . \quad . \quad . \quad . \quad (32)$$

For instance, the observations upon which Brune's mortality table is based (§ 11), give as the most probable number of survivers at the end of a year out of 7943 males of the age 40,

$$l_0=7843\cdot60\,;$$

and as the probable error of this value,

$$\sigma' = 7{\cdot}81.$$

That is to say, it is an even chance that the number of survivers out of 7943 males of the age 40 observed for a year will lie between 7835·79 and 7851·41.

The value 7847 in Brune's table lies between these values.

§ 16.

If we assume $\frac{L'}{L}$ to be identical with p (§ 1), the values of l_0 given by (2) and (26) coincide. But, comparing the equations (8) and (32), we have

$$\sigma' = \sigma\sqrt{1+\frac{l}{L}}. \qquad (33)$$

Hence σ' is always greater than σ, that is to say, if we assume the fraction $\frac{L'}{L}$, which represents the most probable value of the probability of living a year, to be the true value of this probability, we shall get too small a value for the probable error to be expected in the observed number of survivers. It is to be noticed that σ and σ' approach nearer to equality, the smaller l is in comparison with L.

§ 17.

If the period to which the observations are referred is different from a year, all the formulas in §§ 13 to 15 can still be used, as was pointed out in § 5, provided only that L′ denotes the number surviving at the end of that period out of L observed as alive at the beginning of it. But instead of giving the results in that form, it is usual when the same group of persons has been observed during a period of several years, to present the numbers resulting from the observation in the following manner:—

Out of	L persons	of the age	x,	L′ are alive at the end of a year;	
„	M	„	$x+1$,	M′ „	„
„	N	„	$x+2$,	N′ „	„

and so on.

Consequently the question again arises: What is the probability that out of l persons of the age x, l' will survive 1 year; l'', 2 years; l''', 3 years; and so on?

§ 18.

According to (30), the probability P that out of l persons l' will be alive at the end of a year can be expressed by

$$P=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2\left(l'-\frac{L'l}{L}\right)^2}$$

where $$h^2=\frac{L^3}{2lL'(L-L')(L+l)}$$

Just in the same way, the probability, P′, that out of the l' survivers, l'' will survive a 2nd year, is

$$P'=\frac{h'}{\sqrt{\pi}}\epsilon^{-h'^2\left(l''-\frac{M'l'}{M}\right)^2};$$

where $$h'^2=\frac{M^3}{2l'M'(M-M')(M+l')}.$$

Similarly, the probability, P″, that out of the l'' persons, l''' will survive another year, is

$$P''=\frac{h''}{\sqrt{\pi}}\epsilon^{-h''^2\left(l'''-\frac{N'l''}{N}\right)^2};$$

where $$h''^2=\frac{N^3}{2l''N'(N-N')(N+l'')};$$

and so on.

Consequently the probability, Z, of the concurrence of all these events, is

$$Z=\frac{h}{\sqrt{\pi}}\cdot\frac{h'}{\sqrt{\pi}}\cdot\frac{h''}{\sqrt{\pi}}\cdots\cdot\epsilon^{-h^2\left(l'-\frac{L'l}{L}\right)^2-h'^2\left(l''-\frac{M'l'}{M}\right)^2-h''^2\left(l'''-\frac{N'l''}{N}\right)^2\cdots}\qquad(34)$$

Now the most probable values of l', l'', l''', ... are $\frac{L'l}{L}$, $\frac{L'M'l}{LM}$, $\frac{L'M'N'l}{LMN}$, ... respectively; (compare § 13). Put now

$$l'=\frac{L'l}{L}+k$$

$$l''=\frac{L'M'l}{LM}+k'$$

$$l'''=\frac{L'M'N'l}{LMN}+k''$$

&c. = &c.;

whence
$$l' - \frac{L'l}{L} = k$$
$$l'' - \frac{M'l'}{M} = k' - \frac{M'}{M}k$$
$$l''' - \frac{N'l''}{N} = k'' - \frac{N'}{N}k'$$
$$\&c. \quad = \quad \&c.$$

Then the substitution of these last values in (34) gives the formula for the probability that the numbers alive at the end of 1, 2, 3, . . . years, out of l persons observed, will differ from their most probable values, $\frac{L'l}{L}, \frac{L'M'l}{LM}, \frac{L'M'N'l}{LMN}, \ldots$ by the quantities $k, k', k'', \ldots$ respectively,

$$Z = \frac{h}{\sqrt{\pi}} \cdot \frac{h'}{\sqrt{\pi}} \cdot \frac{h''}{\sqrt{\pi}} \cdots \epsilon^{-h^2k^2 - h'^2\left(k' - \frac{M'}{M}k\right)^2 - h''^2\left(k'' - \frac{N'}{N}k'\right)^2 \cdots} \quad . \quad . \quad . \quad (35)$$

Since $h', h'', \ldots$ are independent of k, we can without appretiable error put for $l', l'', \ldots$ their most probable values, $\frac{L'l}{L}, \frac{L'M'l}{LM}, \ldots$ so that we have

$$\left.\begin{aligned} h^2 &= \frac{L^3}{2lL'(L-L')(L+l)} \\ h'^2 &= \frac{LM^3}{2lL'M'(M-M')\left(M+\frac{L'l}{L}\right)} \\ h''^2 &= \frac{LMN^3}{2lL'M'N'(N-N')\left(N+\frac{L'M'l}{LM}\right)} \end{aligned}\right\} \quad . \quad . \quad (36)$$

§ 19.

The probability that the number of survivers at the end of 1, 2, 3, ... years will be contained between the limits $\frac{L'l}{L} \pm k$, $\frac{L'M'l}{LM} \pm k', \frac{L'M'N'l}{LMN} \pm k'', \ldots$ is by (35)

$$= \frac{2h}{\sqrt{\pi}} \cdot \frac{2h'}{\sqrt{\pi}} \cdot \frac{2h''}{\sqrt{\pi}} \cdots \int_0^k dk \int_0^{k'} dk' \int_0^{k''} dk'' \cdots \epsilon^{-h^2k^2 - h'^2\left(k' - \frac{M'}{M}k\right)^2 - h''^2\left(k'' - \frac{N'}{N}k'\right)^2 \cdots} \quad (37)$$

But if we only want the probability that after a given number of years, $n+1$, the number of survivers will differ from its most pro-

bable value by a quantity not exceeding $\pm k^{(n)}$, whatever may have been the deviations in the foregoing years, then $k^{(n)}$ must be the last term of the series $k, k', k'', \ldots$ and the integrations with respect to $k, k', k'', \ldots k^{(n-1)}$ must be taken from 0 to ∞; and only the integration with respect to $k^{(n)}$ from 0 to $k^{(n)}$.

§ 20.

Hence the probability that the number of survivers at the end of 2 years will be contained between the limits $\frac{L'M'l}{LM} \pm k'$ is

$$=\frac{2h}{\sqrt{\pi}}\cdot\frac{2h'}{\sqrt{\pi}}\int_0^\infty dk\int_0^{k'} dk'\,\epsilon^{-h^2k^2-h'^2\left(k'-\frac{M'}{M}k\right)^2} \quad . \quad . \quad . \quad (38)$$

Putting the exponential into the form

$$\epsilon^{-\frac{h^2M^2+h'^2M'^2}{M^2}\left(k-\frac{h'^2MM'k'}{h^2M^2+h'^2M'^2}\right)^2-H^2k'^2}$$

where

$$H^2=\frac{h^2h'^2M^2}{h^2M^2+h'^2M'^2} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

the foregoing integral becomes

$$\frac{2H}{\sqrt{\pi}}\int_0^{k'}\epsilon^{-H^2k'^2}dk' \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and the probable error to be expected in the observed number, l'', is

$$\sigma''=\frac{\cdot 6745}{H\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Again, the probability that the number alive at the end of *three* years will be contained between the limits $\frac{L'M'N'l}{LMN} \pm k''$, is

$$\frac{2h}{\sqrt{\pi}}\cdot\frac{2h'}{\sqrt{\pi}}\cdot\frac{2h''}{\sqrt{\pi}}\int_0^\infty dk\int_0^\infty dk'\int_0^{k''}dk''\,\epsilon^{-\frac{h^2M^2+h'^2M'^2}{M^2}\left(k-\frac{h'^2MM'k'}{h^2M^2+h'^2M'^2}\right)^2-H^2k'^2-h''^2\left(k''-\frac{N'}{N}k'\right)^2}$$

$$=\frac{2H}{\sqrt{\pi}}\cdot\frac{2h''}{\sqrt{\pi}}\int_0^\infty dk'\int_0^{k''}dk''\,\epsilon^{-H^2k'^2-h''^2\left(k''-\frac{N'}{N}k'\right)^2}$$

and putting

$$H'^2=\frac{H^2h''^2N^2}{H^2N^2+h''^2N'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and proceeding as above with (38), this becomes

$$\frac{2H'}{\sqrt{\pi}}\int_0^{k''}\epsilon^{-H'^2k''^2}dk'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Consequently the probable error to be expected in observation of l''' persons is

$$\sigma''' = \frac{\cdot 6745}{H'\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The law according to which these quantities proceed for 4, 5, ... years, is obvious.

§ 21.

The formulas for calculating the probable errors $\sigma, \sigma'', \sigma'''$ .... admit of considerable simplification if we suppose that l is so small in comparison with L, M, N, ... that we may without sensible error put L, M, N, ... respectively instead of the sums $L+l$, $M+\frac{L'l}{L}$, $N+\frac{L'M'l}{LM}$, ... In that case, we get from (36),

$$\left.\begin{aligned} h^2 &= \frac{L^2}{2lL'(L-L')} \\ h'^2 &= \frac{LM^2}{2lL'M'(M-M')} \\ h''^2 &= \frac{LMN^2}{2lL'M'N'(N-N')} \\ \&c. &= \&c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (45)$$

Also from (39) and (42),

$$\left.\begin{aligned} H^2 &= \frac{L^2M^2}{2lL'M'(LM-L'M')} \\ H'^2 &= \frac{L^2M^2N^2}{2lL'M'N'(LMN-L'M'N')} \\ \&c. &= \&c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (46)$$

Consequently from (32), (41), and (44),

$$\left.\begin{aligned} \sigma' &= \cdot 6745\sqrt{\frac{lL'(L-L')}{L^2}} \\ \sigma'' &= \cdot 6745\sqrt{\frac{lL'M'(LM-L'M')}{L^2M^2}} \\ \sigma''' &= \cdot 6745\sqrt{\frac{lL'M'N'(LMN-L'M'N')}{L^2M^2N^2}} \\ \&c. &= \&c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (47)$$

These formulas for $\sigma', \sigma'', \sigma'''$, ... all agree with (8), if we put

in them respectively $\frac{L'}{L}=p, \frac{L'M'}{LM}=p, \frac{L'M'N'}{LMN}=p, \ldots$; and what was said at the beginning of § 5 again holds good. This indeed was to be expected; for the assumption that l vanishes in comparison with L, M, N, ... comes to the same thing as assuming that the probabilities of being alive at the end of the year in question are given.

The suppositions made in this section are usually admissible in practice; as, for instance, in insurance companies, where the number of the members entering at the same age is for the most part small.

(*To be continued.*)

HOME AND FOREIGN INTELLIGENCE.

CALEDONIAN INSURANCE COMPANY.

Commenced Life Business in 1833.

FIFTH SEPTENNIAL INVESTIGATION.

The FIFTH SEPTENNIAL INVESTIGATION into the AFFAIRS of the LIFE DEPARTMENT fell to be made as at 13th May 1871, in terms of the Contract of Copartnery and Act of Parliament, with the object of ascertaining the position of the Company, and of dividing the Surplus which had arisen from the business during the seven years ending on that date.

The liability under Assurance Contracts has been calculated by the Carlisle Table of Mortality, assuming interest at 3 per cent., being the same basis as that on which the former investigations were conducted.

The whole of the "Loading," or addition to the pure premium, was thrown off in the valuation, and thus none of the future profit has been anticipated, nor any of the provision for future expenses encroached upon.

The Annuity Transactions were valued by the Government 4 per cent. tables (Finlaison), Males and Females, and the liability thus brought out was, for greater security, supplemented by a considerable addition thereto; the surplus being carried, in terms of the Contract of Copartnery, to the credit of the Assurance Fund.

The following are the results of the Investigation:—

TOTAL FUNDS after deducting all Outstanding Claims, .	£418,777	15	5
LIABILITY under ASSURANCE and ANNUITY Transactions in force at 13th May 1871, after deducting value of Reassurances, ...	345,537	19	0
SURPLUS	£73,239	16	5

This SURPLUS has been apportioned thus:—

I. To THE ASSURED.—In providing for a REVERSIONARY BONUS OF £1, 7s. PER CENT. PER ANNUM on all Whole Term Policies of five years' duration and upwards in force at 13th May 1871, entitled to participate, AND A REVERSIONARY BONUS OF £1 PER CENT. PER ANNUM on Endowment Assurance Policies of five years' duration and upwards in force at 13th May 1871, entitled to participate,	£43,297	2	2
II. In providing for a Reserve Fund and also a PROSPECTIVE BONUS OF £1 PER CENT. PER ANNUM on all Whole Term Policies of five years' duration and upwards entitled to participate, which may become claims by death before next Division of Profits at 14th May 1878, AND A PROSPECTIVE BONUS OF 10s. PER CENT. PER ANNUM on Endowment Assurance Policies of five years' duration and upwards, entitled to participate, which may become claims by DEATH ONLY before next Division of Profits at 14th May 1878,	19,886	7	4
III. To THE PROPRIETORS, being their Share of the Life Profits in terms of the Contract of Copartnery,	10,056	6	11
SURPLUS as above,	£73,239	16	5

Balance Sheet on the 13*th May* 1871.

LIABILITIES.

Shareholders' Capital £298,075, of which there is paid up				£29,807	10	0
Life Assurance Fund,				398,011	3	11
Annuity Fund,				20,766	11	6
Fire Fund,				117,154	15	5
Re-insurance Fund (Fire),				2,265	15	9
Certain Trust Funds,				2,007	4	7
				£570,013	1	2
Claims under Life Policies admitted but not yet paid,	8,744	7	9			
Outstanding Fire Losses,	2,292	9	3			
Proprietors' Dividends,	1,149	18	6			
Life Department due by Fire,	7,789	6	0			
Due to Agents,	38	1	1			
Sundry Liabilities,	30	19	3			
				20,045	1	10
				£590,058	3	0

ASSETS.

Mortgages over Property within the United Kingdom,	£230,619	13	9
Mortgages over Property out of the United Kingdom,	...	...	
Loans secured over Public Rates,	38,005	0	4
Loans on Company's Policies (Life),	24,581	19	7
Investments—			
British Government Securities,	...	...	
Indian and Colonial Securities,	...	...	
Foreign Securities,	...	...	
Assets carried forward	£293,206	13	8

	£	s.	d.
Assets brought forward	£293,206	13	8
Railway and other Debentures and Debenture Stock, ...	53,312	9	6
Railway Preference Stock, including Purchased Shares of Caledonian Insurance Company, .	31,575	13	3
House Property, ...	37,934	2	0
Landed Estate and Feu-duties, ...	106,554	8	11
Reversions and Life Interests, ...	14,694	1	7
Loans on Personal Security with Life Policies and Company's Shares, ...	10,788	12	4
Agents' Balances, ...	8,692	8	3
Outstanding Premiums, ...	5,344	5	3
Outstanding Interests, ...	10,263	0	1
Cash; on Deposit, ...	...	...	
In hand and on Current Account, ...	8,721	15	8
Other Assets—			
Fire Department due to Life, ...	7,789	6	0
Office Furniture, ...	887	7	2
Stamps, ...	18	8	0
Interest accrued but not due, ...	252	0	0
Sundry Accounts, ...	23	11	4
	£590,058	3	0

The following further particulars are extracted from the Returns under the Fifth and Sixth Schedules of the Life Assurance Companies Act:—

Each Policy was valued separately by finding the value of the Sum Assured, and deducting therefrom the value of the future net premiums, an addition of one quarter year's interest at 3 per cent. being made to the values on account of the claims of the Company being payable three months after proof of death. For Short Period Assurances, a proportion of the premium payable was stated as the liability. The Annuities were also valued separately.

The principles on which the distribution of Profits among the Policy-holders is made:—

The Profits are ascertained every seven years. Five-sixths thereof belong to the Assured, and are divided amongst them by giving to Policies entitled to participate a Bonus at a certain rate per cent. on the sum assured, irrespective of age, for each year's premium paid since the previous division if a Bonus has already been declared on them, and if no Bonus has been added, then for each year's premium paid since the commencement of the risk; a provision being made for the Prospective Bonus which is payable when death occurs between the periods of dividing profits. The remaining sixth of the Profits falls to the Proprietors.

The principles of Valuation and distribution of Profits are determined by the Regulations and Bye-Laws of the Directors.

The Tables of Mortality used in the Valuation were:—

The Carlisle Table for Assurances.

The Government Tables (Finlaison) for Annuities on Single Lives; and

The Carlisle Table for Special Annuities.

A Policy does not share in the Profits until it has existed five

years. At the Division of Profits following the expiry of that period, it is entitled to a Bonus on the sum assured for each year's premium paid since the opening of the Policy, and should it not survive an Investigation, the Prospective Bonus would be payable.

The results of the Valuation:—

1. Total Amount of Profit (including £12,901, 14s. 8d. at credit of Prospective Bonus Account), .	£73,239	16	5
2. Amount of Profit allotted to the Policy-holders:—			
Value of Bonus added to Policies, . .	£43,297	2	2
Prospective Bonus and Reserve, . .	19,886	7	4
	£63,183	9	6

The number of Policies which participated was 1747, and the amount thereof, £721,578.

3. Specimens of Bonuses allotted at 13th May 1871 to Ordinary Policies for £100:—

For all Ages at Entry.	DURATION OF POLICIES.			
	5 Years.	10 Years.	12 Years.	13 Years and upwards.
	£6, 15s.	**£13, 10s.**	**£16, 4s.**	7 Years' Bonus, . **£9, 9s.** A Bonus having already been declared on these Policies.

The whole Bonus is apportioned as a sum to be added to the amount assured. The Policy-holders may, however, on application, draw the cash-values of the additions made to their Policies, or apply the values towards reduction of the premiums for seven years or for life, but no specific portion of the Bonus declared is apportioned beforehand to any one of these methods of disposing of the Bonus, it being optional to the assured to deal with their Bonuses at each Investigation as they may wish.

Consolidated Revenue Account for Seven Years, commencing 14th May 1864 and ending 13th May 1871.

Amount of Funds on 14th May 1864, the beginning of the period, ...				£323,379	7	1
*Half-credit and Alternative Loans, and Outstanding Premiums,—Increase during the Seven years, . .				3,247	3	7
Premiums, after deduction of Re-Assurance Premiums,				290,589	16	9
Consideration for Annuities granted, ...				7,061	12	9
Interest and Dividends, less Income-Tax, . . .				107,351	6	4
Other Receipts—						
Assignation Fees, ...	£83	12	10			
Fines for Revival, ...	74	3	2			
Gain on Investments, ...	1,564	10	6			
				1,722	6	6
				£733,351	13	0

* This entry is required, as no account for these items has hitherto been opened in the Ledger, although taken credit for at the Investigations.

Claims under Policies, after deduction of Sums Re-assured,	£214,950	12	1
Bonuses less Re-assurances,	19,154	17	6
Endowment Assurances matured and paid,	1,849	19	0
	£235,955	8	7
Surrenders, Policies and Bonuses,	10,442	19	9
Annuities,	13,609	13	4
Commission,	11,559	4	8
Expenses of Management,	36,105	19	1
Proprietors' share of Life Profits for Seven years, ending 14th May 1864,	6,900	12	2
Amount of Funds on 13th May 1871, the end of the period, as per Third Schedule,	418,777	15	5
	£733,351	13	0

Summary and Valuation of the Policies (see p. 374.)

Valuation Balance Sheet as at 13*th May* 1871.

Dr.

To Net Liability under Assurance and Annuity Transactions, as per Summary Statement,	£345,537	19	0
To Surplus,	73,239	16	5
	£418,777	15	5

Cr.

By Life Assurance and Annuity Funds, as per Balance Sheet,	£418,777	15	5
	£418,777	15	5

The average rate of Interest at which the Life Assurance Fund of the Company was invested at the close of each year during the period since the last Investigation was:—

At 14th May 1865,	£4	10	0	per cent.
,, 1866,	4	10	2	,,
,, 1867,	4	9	11	,,
,, 1868,	4	9	2	,,
,, 1869,	4	10	5	,,
,, 1870,	4	10	4	,,
,, 1871,	4	11	9	,,

The minimum surrender-value of ordinary whole term Policies and Endowment Assurances on which three full years' premiums have been paid is:—For Non-Participating Policies, twenty-four per cent. of the premiums paid, and for Participating Policies twenty-nine per cent. of the premiums paid, exclusive of the value of any Bonus which may have been added to the Policy; no allowance being made on account of any extra premium which may be payable.

Policies on lives for which an extra is charged on account of health, are valued at the actual age, throwing off the whole of the addition to the net premium; the annual extra being held to cover the additional risk.

Summary and Valuation of the Policies as at 13th May 1871.

Description of Transaction.	Particulars of Policies for Valuation. Number of Policies.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Valuation. Value by the Carlisle Table, Interest 3 per cent. for Assurances. Value by the Government Tables (Finlaison), Interest 4 per cent. for Annuities. Value by the Carlisle Table, Interest 4 per cent. for Special Annuities. Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Net Liability.
ASSURANCES.								
I. With Participation in Profits—		£ s. d.	£ s. d.	£	£	£	£	£
For Whole Term of Life,	2,591	1,076,245 7 4	31,384 16 3	25,010	589,766	455,827	360,850	228,916
Premiums Paid up,	7	2,828 12 0	.. ..	..	1,709	..	..	1,709
Limited Payments,	11	9,642 18 0	381 13 8	309	4,890	2,355	1,807	3,083
Half Premium Plan,	30	18,100 0 0	365 4 5	292	8,471	10,169	8,106	366
Ascending Premiums,	11	4,055 17 0	101 7 5	80	2,230	1,703	1,326	905
Descending Premiums,	3	3,378 10 0	106 0 6	79	1,657	1,299	928	729
Endowment Assurances,	120	23,628 19 3	891 5 10	709	13,335	12,411	10,101	3,233
Joint Lives,	62	19,188 6 6	888 1 7	685	11,758	11,147	8,615	3,143
Longest Liver,	17	11,293 5 0	440 13 5	347	7,024	4,820	3,757	3,267
Extra Premiums payable,	..	.. ..	228 2 10	..	144	..	..	144
Total Assurances With Profits,	2,852	1,168,361 15 1	34,792 5 11	27,511	640,984	499,731	395,490	245,495
II. Without Participation in Profits—								
For Whole Term of Life,	814	381,595 0 0	11,920 11 1	10,265	225,561	150,514	129,143	96,418
Premiums Paid up,	3	3,795 0 0	.. ..	..	2,864	..	..	2,864
Limited Payments,	3	450 0 0	14 17 6	12	201	125	101	100
Half Premium Plan,	10	10,200 0 0	260 4 7	232	5,410	5,403	4,946	464
Ascending Premiums,	19	12,450 0 0	374 9 3	333	7,216	6,357	5,662	1,554
Endowment Assurances,	60	12,271 0 0	502 12 10	445	8,098	5,395	4,755	3,344
Joint Lives,	6	1,400 0 0	60 10 9	50	921	686	569	351
Survivorships,	8	7,300 0 0	103 7 2	86	1,148	997	829	319
Longest Liver,	18	18,100 0 0	440 8 10	381	10,573	5,900	5,104	5,470
Short Period Policies,	25	21,612 0 0	381 2 4	288	239	..	..	239
Extra Premiums payable,	..	.. ..	214 18 6	..	103	..	..	103
Total Assurances Without Profits,	966	469,173 0 0	14,273 2 10	12,092	262,334	175,377	151,109	111,225
Total Assurances,	3,818	1,637,534 15 1	49,065 8 9	39,603	903,318	675,108	546,599	356,720
Deduct Re-Assurances,	..	128,294 5 4	4,133 4 1	3,305	73,446	50,930	40,880	32,567
Net Amount of Assurances,	3,818	1,509,240 9 9	44,932 4 8	36,298	829,872	624,178	505,719	324,153
Adjustment, Addition to Value of Bonuses,	..	.. ..	.. ..	..	..	..	..	618
ANNUITIES.		Annuity.						324,771
Immediate,	59	1,656 12 10	.. ..	..	..	..	..	14,118
On Longest Liver,	3	76 10 0	.. ..	..	..	..	..	1,233
Total Annuities,	62	1,733 2 10	.. ..	..	..	..	..	15,351
Adjustment, Addition to Value of Annuities,	..	.. ..	.. ..	..	..	..	..	5,415
								20,766
Total of the Results,	..	.. ..	44,932 4 8	36,298	829,872	624,178	505,719	345,537

UNION ASSURANCE OFFICE.

Established 1714. *Commenced Life Business* 1813.

The following are extracts from the Board of Trade Returns in reference to the quinquennial valuation and distribution of profits, up to 30 June last. We have ascertained that the Company has issued no other Bonus Report on the occasion.

Balance Sheet on the 30th June, 1872.

LIABILITIES.	£ s. d.	£ s. d.
Shareholders' Capital		30,000 0 0
Life Assurance Fund		723,662 9 7
Fire Fund		559,210 0 4
Mortgage Reserve Fund		8,292 18 8
		1,321,165 8 7
Agency Securities	£1,499 1 8	
Claims under Life Policies admitted but not yet paid	16,608 3 9	
Outstanding Fire Losses	3,325 14 7	
Do. Charges	115 8 3	
Do. Dividends	12,711 13 6	
		34,260 1 9
		£1,355,425 10 4

ASSETS.	£ s. d.
Mortgages on Property within the United Kingdom	682,144 10 10
Loans on the Company's Policies	22,461 0 5
INVESTMENTS:—	
British Government Securities	110,259 10 10
Indian Do.	102,413 19 10
Foreign Do.	6,329 2 0
Railway and other Debentures and Debenture Stocks	242,638 5 3
Railway Shares, Preference and Ordinary	68,095 11 8
House Property	7,513 19 8
Loans on Rates	86,592 0 11
Company's own Shares	2,973 15 9
Loans on Shares of the Company	5,950 0 0
Agents' Balances	6,046 4 1
Outstanding Interest	4,643 10 9
Cash in hand	7,363 18 4
	£1,355,425 10 4

The date up to which the Valuation was made is the 30th June, 1872.

The Tables of Mortality used in the Valuation were the new Experience Table (H^M), published by the authority of the Institute of Actuaries, for all ordinary Insurances, both with and without Profits, and the Carlisle Table for all special policies.

The Rate of Interest assumed in the calculation was Three per cent. throughout.

The whole of the loading has been reserved, being the excess of the Premium payable over the pure Premium, according to the Table used for the Valuation.

The principle of the Distribution of the Surplus is to make a special reserve for future contingencies, and to appropriate four-fifths of the balance to the Policy-holders, and one-fifth to the Shareholders.

The principles upon which the Valuation and Distribution were made were determined partly by the Deed of Settlement, and partly by resolution of the Board of Directors.

The Consolidated Revenue Account for Five Years, from 1st July, 1867, to the 30th June, 1872:

	£	s.	d.
Amount of Funds on 1st July, 1867 (The beginning of the Quinquennial period.)	636,941	3	2
Premiums after deduction of Re-assurance Premiums	414,658	12	7
Interest and Dividends	149,697	18	9
Assignment and other Fees	52	2	0
Profit and Loss on Investments, including Quinquennial Re-valuation of Securities	5,023	12	3
	£1,206,373	8	9

	£	s.	d.
Claims after deduction of sums Re-assured	390,527	19	2
Surrenders	14,477	5	8
Annuities	267	13	1
Commission	30,476	13	8
Expenses of Management	29,222	6	7
Shareholders' proportion of Profits carried to Fire Fund	17,175	10	6
Bad debts, written off	563	10	6
Amount of Funds on 30th June, 1872, the end of the the period, as per Fourth Schedule	723,662	9	7
	£1,206,373	8	9

Summary and Valuation of all the Policies (see p. 377.)

Valuation Balance Sheet of the Society as at 30th June, 1872.

DR.	£	s.	d.
To net liability under Assurance and Annuity transactions (as per Summary and Valuation)	644,484	9	7
„ Reserve for claims which had arisen, but which had not been proved on 30th June	8,957	11	4
„ Surplus	88,565	8	4
	£742,007	9	3

CR.	£	s.	d.
By Life Assurance and Annuity Funds (as per Balance-Sheet)	723,662	9	7
„ Interest accrued	10,285	6	7
„ Amount due on Two-thirds and Half-credit Policies	5,516	14	7
„ Outstanding Premiums not included in Balance Sheet	2,542	18	6
	£742,007	9	3

By the Deed of Settlement the Bonus is ordered to be distributed amongst such persons as shall have participating Policies in force at the time the same shall have been declared, unless such Policies were effected in the then current year, or in the then last preceding year.

The total amount of profit made by the Company in the five years is £88,565 8*s*. 4*d*., in which is included the amount of undivided profit reserved at the last division of profits.

Of this amount £53,247 3*s*. 8*d*. is divided at once amongst 4,035

DESCRIPTION OF TRANSACTIONS.	PARTICULARS OF THE POLICIES FOR VALUATION.				VALUATION by the New Experience (H^M) and Carlisle Tables, Interest 3 per cent.			
	No. of Policies.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Net Liability.
ASSURANCES.		£	£	£	£	£	£	£
I.—With Participation in Profits.								
For whole Term of Life	4,082	2,244,600	70,423	56,729	1,296,267	945,838	752,431	543,835
Other Classes:—								
Life by Limited Premiums	37	24,562	1,190	1,013	11,958	7,618	6,417	5,541
„ Varying Premiums	10	13,135	278	222	7,740	3,863	3,091	4,650
Paid up Policies	25	10,725	...	...	7,354	...	...	7,354
Joint Life Policy	1	216	6	4	164	49	33	130
Life of Last Survivor	1	964	8	7	709	64	55	654
Extra Premiums Payable	...	...	118	...	...	...	...	...
Total Assurances with Profits	4,156	2,294,202	72,023	57,975	1,324,192	957,432	762,027	562,164
II.—Without Participation in Profits.								
For whole Term of Life	388	243,210	8,221	7,146	146,146	97,346	84,286	61,859
Other Classes:—								
Endowment Assurances	123	30,864	1,452	1,269	21,504	13,403	11,652	9,852
Life by Limited Premiums	1	150	6	4	83	32	25	58
Short Period Insurances	16	14,070	386	270	679	960	615	64
Life by Varying Premiums	34	25,550	662	530	14,019	9,447	7,558	6,462
Paid up Policies	3	2,250	...	...	1,093	...	...	1,093
Joint Life Insurances	24	6,300	294	235	4,009	3,448	2,752	1,257
Life of Last Survivor	9	3,150	49	38	1,218	936	722	496
Survivorships	10	9,200	184	124	1,157	1,261	874	283
Assurances against Issue	3	13,000	...	...	520	...	...	520
Extra Premiums Payable	...	...	59	...	...	...	...	...
Total Assurances Without Profits	611	347,744	11,313	9,616	190,428	126,833	108,484	81,944
Total Assurances	4,767	2,641,946	83,336	67,591	1,514,620	1,084,265	870,511	644,108
Being the *net* amount, &c., of the Assurances, as the Re-assurances enumerated in Schedule 6 (see page) have already been deducted from the corresponding amounts at each age, and under each class.								
ANNUITIES.								
Immediate	2	49	...	...	376	...	...	376
Total of the results			83,336	67,591	1,514,996	1,084,265	870,511	644,484

Policies for the sum of £2,049,126, and £22,006 8*s.* 9*d.* retained to form a special reserve as stated above.

The Bonus allotted to every Policy for £100, which has been in force for five years and upwards, is £5; the equivalent amounts of cash will be found in column A, and the equivalent reductions of Premium for five years only in column B, of the following table:—

Age at Entrance.	NUMBER OF YEARS FOR WHICH POLICY FOR £100 HAS BEEN IN FORCE.												Age at Entrance.
	5.		10.		15.		20.		25.		30.		
	A	B	A	B	A	B	A	B	A	B	A	B	
	£ *s.* *d.*	*s.* *d.*	£ *s.* *d.*	*s.* *d.*	£ *s.* *d.*	*s.* *d.*	£ *s.* *d.*	*s.* *d.*	£ *s.* *d.*	*s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	
20	1 2 11	5 1	1 5 8	5 9	1 8 8	6 5	1 12 0	7 2	1 16 1	8 2	2 0 11	0 9 4	20
30	1 8 8	6 5	1 12 0	7 2	1 16 1	8 2	2 0 11	9 4	2 5 10	10 7	2 11 0	0 11 11	30
40	1 16 1	8 2	2 0 11	9 4	2 5 10	10 7	2 11 0	11 11	2 16 11	13 6	3 3 1	0 15 5	40
50	2 5 10	10 7	2 11 0	11 11	2 16 11	13 6	3 3 1	15 5	3 9 6	17 9	3 16 4	1 1 2	50

The average rate of interest at which the Life Assurance Fund was invested at the close of each year of the Quinquennial Period was as follows:—

Year ending 30th June.	Average rate.
	£ *s.* *d.*
1868	4 11 9
1869	4 12 6
1870	4 12 10
1871	4 13 2
1872	4 12 7

The following is a table of minimum values allowed by the Society for the surrender of its Whole-Life Policies; it has no fixed rule for the purchase of Endowment Assurances, each case being valued upon its own merits:

Sum Assured £100.

Age at Entrance.	NUMBER OF YEARS IN FORCE.								Age at Entrance.
	5.	10.	15.	20.	25.	30.	35.	40.	
	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	
20	2 13 2	5 10 9	8 14 2	12 4 6	15 19 0	20 0 0	24 3 0	28 11 6	20
25	3 0 1	6 6 4	9 19 9	13 17 6	18 2 0	22 8 8	27 1 0	32 0 3	25
30	3 9 6	7 6 7	11 8 3	15 17 0	20 7 11	25 5 0	30 9 0	36 0 0	30
35	4 1 6	8 8 0	13 1 10	17 18 0	23 0 8	28 10 10	34 8 3	40 3 0	35
40	4 12 4	9 12 9	14 15 5	20 5 3	26 3 0	32 8 6	38 11 2	44 4 5	40
45	5 11 1	11 4 11	17 6 4	23 16 8	30 15 8	37 11 5	43 16 11	48 19 8	45
50	6 4 5	12 17 3	19 19 9	27 11 8	35 0 1	41 17 4	47 9 9	51 9 7	50

The Society has no business at other than European rates, and in valuing Policies on unhealthy lives the rated age is that at which the valuation is made; if, however, the Policy is valued for surrender, the true age is taken.

CORRESPONDENCE.

ON THE EMPLOYMENT OF THE "INSTITUTE OF ACTUARIES' LIFE TABLES" IN FINDING THE VALUE OF AN ANNUITY ON THE LAST SURVIVOR OF THREE LIVES.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—It may be well to point out that notwithstanding the terms of "Simpson's formula" are so conveniently tabulated in the Institute Tables, the employment of these tables in the mode recently suggested to you involves the introduction of an error from which the ordinary formula is free. The formulas may be written as follows.

Simpson's:—

$$a_{\overline{xyz}} = (a_x + a_y - a_{xy}) + (a_x + a_z - a_{xz}) - (a_x + a_w - a_{xw}).$$

The ordinary formula:—

$$a_{\overline{xyz}} = (a_x + a_y + a_z) - (a_{xy} + a_{xz} + a_{yz}) + a_{xw}.$$

Both are equally tainted with the error of the assumption (expressed at the close of each), that because $a_w = a_{yz}$, therefore $a_{xw} = a_{xyz}$; but the difference in the notation of these two formulas will be found (upon cancelling whatever is common to both) to be that Simpson's third term has a_w instead of the a_{yz} in the second term of the common formula. These are theoretically equal, but upon converting Simpson's third term in gross into figures by adopting for $a_{\overline{xw}}$ a value tabulated in the Institute Tables, a new error is here introduced through the incorporation of a value for a_w which is only a more or less close approximation to the true and known a_{yz} instead of being exactly equal to it as contemplated in the formula itself; and to this circumstance the difference ·0935 between the results of the two calculations on p. 267 is traceable. Thus, in extracting 20·8569 or $a_{\overline{30,56}}$ (that is, $a_{30} + a_{56} - a_{30,56}$) the value 11·7242 of a_{56} replaces the known true value 11·8177 of $a_{40,50}$; and their difference is the ·0935 just mentioned. To guard against the introduction of this difference, which will produce sometimes an excess and sometimes a deficiency, the notation of Simpson's formula, if for use with the Institute Tables, may be altered to

$$a_{\overline{xyz}} = a_{\overline{xy}} + a_{\overline{xz}} + a_{xw} - (a_x + a_{yz})$$

Working in this manner with the given example, we have

		$a_{\overline{30,40}}$	22·2274
		$a_{\overline{30,50}}$	21·2947
		$a_{30,56}$	10·7347
			54·2568
a_{30}	19·8674		
$a_{40,50}$	11·8177		31·6851
As by the common formula p. 267			22·5717

The facilities afforded by the last-survivor values are indeed thus reduced, but the avoidance of the new error is a matter of greater importance.

This error is, it will have been seen, quite extra to the well-known error involved in the acceptance of a_{xw} for a_{xyz}, and which, as already pointed out, equally affects Simpson's formula and the common one. For the lessening of this error, Milne (Article 529) has given estimates. He would have taken for w neither the fractional age at which a single life annuity exactly equals an annuity upon two joint lives aged y and z; nor the nearest integral age; nor yet (in this example) the next higher integral age; but he would have put for w, 55·8. Thus,

$$\begin{array}{lr@{\qquad}lr} a_{55} & 12{\cdot}0938 & a_{55} & 12{\cdot}0938 \\ a_{56} & 11{\cdot}7242 & a_{40,50} & 11{\cdot}8177 \\ \hline & {\cdot}3696) & & {\cdot}2761({\cdot}7 \end{array}$$

Milne would increase 55·7 to 55·8 for w.

$$\begin{array}{r@{}l} a_{30,55}= & 11{\cdot}0378 \\ a_{30,56}= & 10{\cdot}7347 \\ \hline & {\cdot}3031 \\ & {\cdot}8 \\ \hline & {\cdot}2425 \\ a_{30,55}= & 11{\cdot}0378 \\ \hline \end{array}$$

Milne would put $a_{30,55\cdot8}$ or 10·7953 for $a_{30,40,50}$,
instead of the $a_{30,56}$ or 10·7347 put for ,, at p. 267.

So he would, in effect, have added ·0606
to the result on p. 267 22·5717

and have given 22·6323 for $a_{\overline{30,40,50}}$.

This is very close to the value, 22·616, which is the mean of those you have determined by Mr. Woolhouse's formula.

I remain, Sir,
Your obedient servant,

EDWARD SMYTH.

8 *Mostyn Terrace, North Brixton,*
17 *Feb.* 1873.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

On a general Formula for the value of present or future Benefits, whether free or burdened with charges; and on the application of the Formula to determining the Surrender Values of Life Policies. By James R. Macfadyen, *of the Legal and General Life Assurance Society.*

[Read before the Institute, 31 March 1873.]

WITH the exception perhaps of extra premium, there is, it seems to me, no actuarial point in which the practice of assurance offices is so empirical as that of the allowance to be given for the surrender of a policy. The usual custom, so far as I am aware, is to find the value of the policy (generally by the table of mortality and interest used in calculating the reserve) and from that value to deduct a certain proportion, handing over the remainder to the policyholder. The amount of this deduction is purely arbitrary, and were it not for the equalizing effects of competition, would show very startling diversities; since, even with this counteracting effect, the differences in practice can hardly help striking anyone who has examined the amounts given as surrender value by the various offices interested in large reassurance cases which from any reason do not follow the rule of the parent company. In fact, making any return whatever under discontinued policies is the

result of that struggle for existence called competition; and like all things growing from this root, the practice has been moulded much more by the pressure of circumstances than by theoretical considerations. Consequent on this method of growth we find extraordinary and indefensible anomalies imperatively calling for rectification. Further it seems to me, looking at the matter as a purely practical question and not one of actuarial fitness at all, there is no point more needful of examination. As things are at present, nobody can say positively whether as its proportion of the value the assurance office retains too little or too much. If the former, the sooner a practice resulting in loss is reformed, the better. If the latter, looking at the modern principle of conducting life assurance business, it is surely expedient that the policyholder receive all that he can fairly lay claim to;—a thing all the more necessary, as I consider the smallness of the return made on surrender, one of the chief deterrents to the spread of life assurance, and a fertile source of disgust to policyholders. No doubt in many cases, they expect to be literally assured for nothing, and expectations thus founded must of course be disappointed; but, if the value offered fall below that which could reasonably be allowed, this is no argument why it should not be increased so far as justice will permit of. Another proof that the subject needs consideration may be found in the recent tendency of the provident public to requiring in advance statements as to the amount to be obtained on withdrawal from the office they may assure in;—a tendency which the plan of guaranteed surrender values, non-forfeiture schemes, and the like devices, are intended to meet. The objects I have set before myself in the following paper, therefore, are—*first,* to attempt to discover a method of calculating surrender values which will have at least a basis defensible in theory; and *second,* from the theory to deduce methods of working readily applicable in every-day practice.

In the event of the cancelment of a life policy, there are, I think, only two arguments that can be brought in support of the course usually followed of making some deduction in the value allowed from that reserved as liability. These are—first, that the rate of interest employed for reserve purposes is dictated by prudential motives and cannot be considered as an investment rate; and second, that in that surrender there is a selection exercised against the society, as mainly those who are more than average lives will relinquish their policies, and thus the mortality rate experienced by those remaining will be intensified by the with-

drawals. In the course of our analysis we shall find the first argument crop up, so it can be more fitly discussed hereafter. The second, however, can be dealt with at once. There can, I think, be no doubt of the reality of the selection exercised against the office in withdrawals. Mr. Sprague, in his very valuable paper in the fifteenth volume of the *Journal*, page 328, has clearly shown its existence from what I may call its reflection on the death rate of those left behind. The first problem then that presents itself is to estimate the deduction to be made on this account. We shall find in the more general question as to the theory of the subject some light thrown on this point, so we shall lay it aside for the moment and pass on to discuss that broader topic.

Looking at the value of a policy in the widest possible aspect, viz., as the price a purchaser could give for a benefit (not necessarily dependent on life), the following general expression will suit every case. I say "could give", because I intend the word "price" to include not only the present selling price in the market, that is, the sum it would fetch as an investment, but also its present selling price to the person who is under the obligation to provide the benefit, or, in other words, its surrender value; its present cost price, that is, the liability under it; and its original cost and selling prices, or, in other words, the liability when the benefit was granted, and the price at which it was granted respectively. The price that can be given by a purchaser of any benefit is the *amount* of that benefit less the cost of the annuity-due to provide

1st, the interest on the assumed outlay till the benefit becomes payable;

2nd, the premiums to guard against the contingencies that might prevent the purchaser obtaining the benefit bought;

3rd, the burdens on the benefit.

As applications of this general expression let us take the following from the infinite number of possible cases.

The value or cost of a reversion of 1 payable absolutely on the death of a person now aged $(x+n)$, or, in other words, a single-payment policy, is the amount of the benefit less an annuity-due of the interest on the outlay. (The diminutions under headings 2 and 3 are *nil.*) That is,

$$\text{Value (or cost) of Reversion} = {}_{n|}\mathrm{V}_x = {}^{n}\mathrm{A}_x = 1-(1-v)\,{}^{n}\mathbf{a}_x.$$

The value of a similar reversion on the same life, but subject to the annual payment, now due, of P_x, that is, the value of a whole-life policy by annual premiums, is the amount of the benefit less the cost of an annuity-due of the interest and the burden. (The diminution under heading 2 is *nil.*) That is,

$$\text{Value (or cost) of Reversion} = {}_{n|}V_x = 1 - (1 - v + P_x)\,{}^n\mathbf{a}_x.$$

The value of a reversion of 1 on the same life, payable however only if another person now aged $(y+n)$ be alive at the time of his death, and subject during the joint lives to a payment of P^1_{xy}, or, in other words, the value of a survivorship assurance policy opened n years ago and payable if a life then aged x predecease a life then aged y, is the amount of the benefit, less the cost of an annuity-due to provide

1st, the interest on the outlay during the joint lives;

2nd, the annual premium to insure against the person aged $(y+n)$ dying before the one aged $(x+n)$, or ${}^nP_{x\overset{1}{y}}$;

3rd, the burden of P^1_{xy}.

That is,

$$\text{Value of Reversion} = {}_{n|}V^1_{xy} = 1 - \{1 - v + {}^nP_{x\overset{1}{y}} + P^1_{xy}\}\,{}^n\mathbf{a}_{xy}.$$

The price (or value) of the sum of 1 payable if the life aged $(x+n)$ die within the next $(t-n)$ years, but subject to an annual payment during the said period of $P_{|t}A_x$, or the value of a temporary assurance policy for t years by annual premiums opened at age x, the present age being $(x+n)$, is the amount of the benefit, less the cost of the annuity-due to provide

1st, the interest on the outlay during the risk period;

2nd, the annual premium to furnish the benefit if the life survive age $(x+t)$, or ${}^nP_{x\overset{1}{\overline{t|}}}$;

3rd, the burden of $P_{|t}A_x$.

That is,

$$\text{Value (or price) of benefit} = 1 - \{1 - v + {}^nP_{x\overset{1}{\overline{t|}}} + P_{|t}A_x\}\,{}^n\mathbf{a}_x.$$

The price (or value) of a reversion on a life aged x, commencing at 1 and increasing 1 per annum till his death (that is, the present value of an increasing assurance), is the utmost sum that the purchaser of the reversion can receive, less the cost of the annuity-due to provide

1st, the interest on the outlay of $(\omega - x)$, ω being the limiting age;

2nd, the annual premium on an assurance of $(\omega-x-1)$ diminishing 1 per annum till death, say $\mathbb{P}$;

3rd, *nil.*

Value of benefit $=\omega-x-\{(1-v)(\omega-x)+\mathbb{P}\}\mathbf{a}_x=\dfrac{R_x}{D_x}$.

It would be easy to multiply further examples: as a final illustration, however, let us take the present value of an annuity.

Griffith Davies's formula for the price of an annuity, it will be evident, is a case of the same general expression, as the interest on the outlay and the replacement of capital (headings 1 and 2, with 3 *nil*) are provided for by him. However, as a matter of curiosity, though the resulting formula cannot be considered elegant, let us find the price of an annuity in a manner similar to that in which the increasing assurance value given above is arrived at.

The price that can be given for an annuity of 1 payable during the lifetime of a person aged x, or the value of an annuity of 1 at age x, is the utmost sum that the purchaser of the annuity can receive, less the cost of the annuity to provide the interest on his outlay and less the cost of the assurance to replace the capital when x dies.

That is, the sum possible to be received, or $\omega-x$,

less 1st, the annuity-due to provide the interest on his outlay, that is (since his capital is yearly partially replaced by the annuity payments), an annuity-due commencing at $(1-v)(\omega-x)$ and falling $(1-v)$ per annum till x's death, or $(\omega-x)(1-v)+(1-v)\dfrac{(\omega-x)N_x-S_x}{Dx}$, or say $\mathbf{a}$;

less 2nd, the single payment to replace the capital on x's death, that is, an assurance commencing at $(\omega-x)$ and falling throughout life 1 per annum, or $\dfrac{(\omega-x+1)M_x-R_x}{D_x}$, or say $\mathfrak{A}$.

That is, the price of annuity

$$=(\omega-x)-(1-v)(\omega-x)-(1-v)\frac{(\omega-x)N_x-S_x}{D_x}-\frac{(\omega-x+1)M_x-R_x}{D_x}$$

$$=(\omega-x)-\mathbf{a}-\mathfrak{A}=a_x=\frac{N_x}{D_x}.*$$

Similarly the values of other benefits can be arrived at.

* If $\omega-x$ be taken as 1, these formulæ become respectively

For the increasing assurance, $1-\{(1-v)+\mathbb{P}\}\mathbf{a}_x$,

and for the annuity, $1-\mathbf{a}-\mathfrak{A}$.

I have dwelt at some length on this expression, as it presents at one view the values (according to any meaning of the word) of benefits of the most varied kind. Whether the value to be found is the price as an investment, the amount to be reserved as liability, the surrender value, or the premium to be charged, the problem is reduced in each case to the question of the rates of interest and mortality and the loading to be employed in the different parts of the formulæ resulting from the general expression I here give. The special value taken up in this paper is that amount which may be paid for the benefit by the original granter of it: but it is impossible altogether to dissociate this meaning of the word "value" from the other senses in which it may be used, and more especially from that in which it represents the sum that could be given for the benefit by anyone not under obligation to supply it. So though any remarks that may be made will, unless when otherwise stated, have a primary reference to the price that may be given for the cancelment of a policy, they will have a secondary application to values of other kinds also.

Examining the general expression before given, it will be seen that in every surrender value paid, an assurance office is, whether it likes it or not, in reality becoming a dealer in annuities,—purchasing annuities from itself at the expense of its policyholder, and recouping itself for the amount paid away on the policy becoming a claim; and thus no assurance, whether nominally surrendered or not, can in this way of regarding the matter really be considered as cancelled. It simply changes hands, and is now continued for the benefit of the office and not of its former owner. Obviously, then, selection in discontinued assurances simply means that those persons on whose lives the annuities are granted and reversions depend, will, on the whole, live longer than those who do not sell their policies; and it follows that the force of selection can be met in precisely the same manner as the purchaser of a reversion may protect himself against unusual longevity in the *cestui que vie*. Taking then the purchasing annuity price given by Mr. Jellicoe (*Assurance Magazine*, vol ii, page 149) as a fair one, we find in, say, the case of an ordinary whole life policy by annual premiums, that in the formula

$$1-\{(1-v)+P_x\}\,{}^{n}\mathbf{a}_x$$

if we substitute for the pure annuity the Carlisle $3\frac{1}{2}$ per-cent annuity, we shall have fully allowed for selection. It is usual in surrender-value calculations to consider P_x as the pure premium,

and so we shall for the present take it. If then we apply our reasoning to the case of an office valuing on a 3 per-cent basis, it might at first sight seem that a surrender value *could* be allowed of

$$1-\{\overset{3\%}{\overline{1-v}}+\overset{3\%}{P_x}\}\overset{\text{Car. }3\frac{1}{2}\%}{{}^{n}\mathbf{a}_x}.$$

Now we know the full liability value under the Carlisle Table is

$$1-\{\overset{3\%}{\overline{1-v}}+\overset{3\%}{P_x}\}\overset{\text{Car. }3\%}{{}^{n}\mathbf{a}_x}$$

—an amount ($\overset{\text{Car. }3\frac{1}{2}\%}{{}^{n}a_x}$ being less than $\overset{\text{Car }3\%}{{}^{n}a_x}$) smaller than the preceding. So we are met at the outset by the paradox that, after making allowance for the selection in surrender values, a greater sum can be given away than the actual amount held in hand; and our wonder at this extraordinary result would not be lessened when, in examples, we find that the surrender value thus proposed to be paid away, is sometimes five or six times as great as the total premiums received. But, it may be urged, since a surrendered policy really means a policy not surrendered but continued by the office on its own account, this extraordinary value represents not merely the present worth of the policy in the Society issuing it, but the worth of the future profit that will be made on the transaction by this Society, if it chooses to purchase its annuity not from itself but from another Company selling on the less onerous terms of 3½ per-cent. Let us examine if this is so or not. For illustration take two offices. The first, (A), deals in assurances alone, which it sells on a 3 per-cent basis. The second, (B), grants both assurances and annuities on a 3½ per-cent basis, the annuities without any loading for expenses, and the assurance premiums having an addition identical in *amount* with that charged by Office (A), and therefore possible to be disregarded in the following comparison. The expenses in each society are assumed to exactly swallow up the loading, and the mortality is also taken as the same in both. Consider also Office (A) as able of itself to realize 3½ per-cent interest on only that part of its investments which exceeds the 3½ per-cent premium, while Office (B) realizes 3½ per-cent throughout. A policyholder assures at age x in Office (A), which, in order to realize 3½ per-cent on *all* its investments, reassures the full amount in Office (B), and thus in the language of the Stock Exchange instead of a jobber becomes a broker,—a broker without his percentage, however, as we shall assume (B) a non-commission-paying office. At the end of n years the policyholder wishes to surrender his assurance in Office (A),

which, let us say, undertakes to let him have not merely the immediate value of his policy, but all the profit that has been, or will be, made on the transaction. What is the expression for the sum he will receive? First, on account of the past he will get the accumulated difference between the 3 and the $3\frac{1}{2}$ per-cent premiums, or

$$\{\overset{3\%}{P}_x-\overset{3\frac{1}{2}\%}{P}_x\}\frac{\overset{3\frac{1}{2}\%}{N_{x-1}-N_{x+n-1}}}{D_{x+n}}.$$

Secondly, if the expression

$$1-\{\overset{3\%}{\overline{1-v}}+\overset{3\%}{P}_x\}\,{}^{n}\overset{3\frac{1}{2}\%}{\mathbf{a}}_x$$

includes the future profit as well as the present value and nothing else, he will further get the sum represented by this formula. Can he do so?. It is obviously impossible, if Office (B) likewise undertook to disgorge everything realized in the way of profit, that, on surrender of his policy, the policyholder can be worse off had he from the first directly insured in (B) than indirectly in it through (A). The benefit must be identical. From Office (B), had he insured in it at age x, he would at age $(x+n)$ receive nothing on account of the 3 per-cent premium having been paid. As, however, he would himself have saved and accumulated its excess over the $3\frac{1}{2}$ per-cent premium, this item is the same in each case, and consequently may be rejected. For his direct assurance with (B) he could only receive the value of his policy or

$$1-\{\overset{3\frac{1}{2}\%}{\overline{1-v}}+\overset{3\frac{1}{2}\%}{P}_x\}\,{}^{n}\overset{3\frac{1}{2}\%}{\mathbf{a}}_x,$$

—an amount which being considerably less than

$$1-\{\overset{3\%}{\overline{1-v}}+\overset{3\%}{P}_x\}\,{}^{n}\overset{3\frac{1}{2}\%}{\mathbf{a}}_x,$$

the sum that (A) might seem able to give, shows that the latter expression is incorrect. In order to discover where the incorrectness lies, let us analyze the amount apparently able to be offered by (A). Office (A) gives the value of its reassurance in (B) + the value for the rest of life of the difference in the rate of interest realized and the rate it is contented with,—the value for the rest of life of the difference in the premium it demands from that which is required. Putting this into a formula it becomes

$$1-\{\overset{3\frac{1}{2}\%}{\overline{1-v}}+\overset{3\frac{1}{2}\%}{P}_x\}\,{}^{n}\overset{3\frac{1}{2}\%}{\mathbf{a}}_x+\{\overset{3\frac{1}{2}\%}{\overline{1-v}}-\overset{3\%}{\overline{1-v}}\}\,{}^{n}\overset{3\frac{1}{2}\%}{\mathbf{a}}_x-\{\overset{3\%}{P}_x-\overset{3\frac{1}{2}\%}{P}_x\}\,{}^{n}\overset{3\frac{1}{2}\%}{\mathbf{a}}_x;$$

that is,

$$1-\{\overset{3\%}{\overline{1-v}}+\overset{3\%}{P}_x\}\,{}^{n}\overset{3\frac{1}{2}\%}{\mathbf{a}}_x.$$

The impossible excess over the 3½ per-cent value, it will be thus seen, arises from the fact that Office (A) assumes 3 per-cent interest only in its calculation, while 3½ per-cent is actually realized. The giving the policyholder on surrender the benefit of a 3½ per-cent annuity, is quite inconsistent with itself only taking 3 per-cent interest on the transaction. It must realize 3½ per-cent if it is to give it; and the very fact of its buying an annuity at 3½ per-cent and allowing its policyholder the benefit of the rate it purchased at, compels the assumption in the other part of the formula of an interest rate not less than 3½ per-cent. Since, however, it realizes 3½ per-cent interest and not 3 per-cent, it can, on the other hand, afford to take the future premium at the former rate.

Having now in the equation

$$_{n|}\mathrm{V}_x = 1 - \{1 - v + \mathrm{P}_x\}\,{}^{n}\mathbf{a}_x$$

considered the effect of conflicting interest rates, let us next proceed to examine the results of the introduction into the various parts of the expression of diverse mortality tables. Let a dash over a symbol signify that in that symbol the mortality table involved is that which those surrendering their policies obey, while unaccented life symbols show a mortality corresponding to that of the general body of assured lives. (This of course includes the other mortality also.) Now persons withdrawing from the office, say at age $(x+n)$, will have on the average a vitality superior to that of the lives remaining who were assured at the same time as themselves or previously; but it does not follow that this vitality must be as great as that of the recent medically selected entrants; and thus it is possible to suppose a table of mortality such that ${}^{n}a_x$, affected as it is by the artificial vitality of these and future entrants, is not less than ${}^{n}a'_x$. Even if ${}^{n}a_x$ were greater than ${}^{n}a'_x$, it would not be possible to allow the lives having this latter mortality the benefit in the surrender of their policies of their inferior vitality, as from the very nature of the case the value given must be an average one, and, therefore, not greater than the amount reserved as liability. If, however, a solvent office could be found to reassure in, (I suppose I must use this word as "reannuitize" is not known to the English language), which was willing to grant annuities on the terms ${}^{n}a'_x$, then, if the annuity was actually bought, Office No. 1 retaining the cost of future expenses and having security for the periodical appearances of its former policyholder, might use the smaller annuity ${}^{n}a'_x$ in its surrender values.

Summarizing then, interest and mortality both being supposed variable in the expression

$$1-\{\overline{1-v}+P_x\}\,{}^n\mathrm{a}_x,$$

taking first the unpractical case of an annuity being actually bought from an annuity office, not merely the annuity but the interest part of the surrender-value formula should be by the same rates as those founded on by the Annuity Society; or if different, at least not be calculated by tables that will bring out a higher value than would be arrived at by taking the annuity office's rates. Since, however, partly in consequence of no society wishing to continue a transaction in another company without further benefit to itself; and partly in consequence of the difficulty in keeping the original policyholder in view after he has sold his interest in his reversion, this form will never arise in practice, all that we need take into consideration is the effect of the introduction of various rates of mortality and interest into the formula, when the office originally insuring undertakes also the annuity risk. In this case, if the standard assumed be the rates employed in the calculation of the insurance reserve, then the annuity selling price must not be founded on interest and mortality rates greater than in that reserve. Or, on the other hand, if the standard taken be the annuity, the interest for the rest of life must be not less than that employed in that annuity. A deficiency in one of the rates might be compensated for by an excess in the other, so all that can be absolutely laid down is that the surrender value of the policy cannot be greater than the amount reserved as liability against it. If as great, it may simply mean that the office grants annuities on more favourable terms than it is willing to grant assurances; an anomaly, no doubt, but one very prevalent indeed.*

I have dwelt at some length on this part of the subject, because it is essential to the carrying of the theory of surrender values into practical working. Even looking at the question

* Mr. Jellicoe, in a paper in the *Assurance Magazine*, volume viii, page 310, considers the cheaper price of annuities than assurances to be caused by Government competition in the former. No doubt this has something to do with it, but I am disposed to think the principal reason is that annuity transactions are a large and *distinct* branch of business, on which no bonuses are paid to bondholders. "With-profit" policyholders are content to pay more than is absolutely required for the risk, in consideration of the bonuses received; and as profit policies form the great bulk of assurance business, the non-profit class does not receive the same attention. Were profit policies done away with, I think the rates for non-profit assurances would from the effects of competition very soon approximate to those for annuities.

from the standpoint of the purchase of reversions as investments, the analysis is useful, as it shows the limitations caused in one part of the formula by the introduction of interest or mortality rates in another part.

We have now considered the principle proposed to be applied in the calculation of surrender values. It is as follows;—Consider each policy cancelled in this way as a reversion purchased, using in the various parts of the formula by which the price paid for is arrived at, tables of interest and mortality consistent, not only with each other, but with the terms on which the office is willing to grant annuities. It is no objection to this for a Society to say that it does not wish to deal in annuities at all; as, if it allows surrender value for its policies, it, *nolens volens,* issues annuities. Neither can it urge that since it must sell them it can fix its own terms for them, and these terms are such as to bring out surrender values on the present mode of practice. Since by doing this, as was said at the beginning of this paper, very great anomalies will be produced. As for instance it may, and indeed will happen, that in policies of long duration, the value offered by the Society is less than can be given by a private purchaser using Mr. Jellicoe's formula. In such a case, if the actuary of the Company is first asked as a matter of office routine to quote the sum that will be given for the surrender of the policy; and afterwards his opinion be requested as to the amount a private purchaser can offer for it; it will, I think, be rather difficult to convince the holder of the policy that the office issuing it, ought to give less for it than one who was not a dealer in reversions.

Let us now proceed to the application in practice of the foregoing principle of finding surrender values, and as a preliminary to this, it is expedient to analyze the rates introduced into the formula for a reversionary interest, given in the paper before referred to, and also those proposed by Mr. Sprague as their substitute in volume xiv of the *Journal of the Institute,* page 417. The remarks I shall find it necessary to make are applicable to every case discussed by these gentlemen of the general expression I have already given. But, in order to have the advantage of concentration, let us look at, say, the formula for a reversion payable on the death of a person aged $(x+n)$ if another aged $(y+n)$ be then alive. Substituting for the annuity on the joint lives that on the single life and for the survivorship premium, the whole life premium on age x, this is, of course, the same as the expression for an ordinary policy value. Mr. Jellicoe, in order to

arrive at the price to be given by a person who wished to buy an isolated reversion such as this, employs the following

$$1-\{\overset{5\%}{\overline{1-v}}+\overset{\text{Office rate}}{{}^{n}P^{\;1}_{xy}}\}\,\overset{\text{Car. }3\frac{1}{2}\%}{{}^{n}\mathbf{a}_{xy}}.$$

This expression is founded in every particular on market rates. It consequently is thoroughly consistent with the principle on which it is based;—viz. placing the purchaser of every reversion in the position requisite for him if he had that reversion only. Mr. Sprague, to use his own words, objects "to introducing the "idea of the purchase of *isolated* reversions." He further takes exception to Mr. Jellicoe's formula for an absolute reversion on other grounds, which as before I shall quote, "viz., that it sup- "poses an annuity to be purchased to provide interest on the "outlay until the reversion falls in, whereas such an annuity is "never actually purchased in practice. This being the case, if the "value of a reversion is estimated by Mr. Jellicoe's formula, "allowing 5 per-cent interest to the purchaser, while the annuity "is calculated at 3½ per-cent only, the purchaser of a large "number of reversions, such as a Reversionary Society, would "make a profit on the assumed grant of these annuities in "addition to the 5 per-cent. It therefore appears to me much "more satisfactory to include the whole profit of the transaction "under one aspect, and to assume that the reversions are on the "average bought on such terms as to yield a higher rate of "interest."

As Mr. Sprague himself says of Messrs. Jellicoe and Tucker, I also may say of him,—that anything that is recommended by his high authority calls for the most careful consideration, and any actuary who dissents from his views should be prepared to state fully and clearly his reasons for such dissent. I shall therefore do what I can to show why I take exception, not to his general reasoning quoted above, but to his formulæ for contingent reversions and reversionary life interests, because in my opinion they do not carry out that reasoning consistently. The formula given by Mr. Sprague for, say, the value of the contingent reversion previously referred to is

$$1-\{\overset{6\%}{1-v}+\overset{\text{Office rate}}{{}^{n}P^{\;1}_{xy}}\}\,\overset{\text{Car. }6\%}{{}^{n}\mathbf{a}_{x}}.$$

This formula, since it contains a profit distinct from the 6 per-cent interest, is at variance, as I understand it is intended to be, with the principle laid down for an absolute reversion. I can see

no reason if, in the case of the latter, it is much more satisfactory that the whole profit of the transaction should be included under one rather than under several heads, why in the contingent reversion the whole profit should not also be included under the one heading of interest. There is a distinction made where I think there is no difference. The main object of putting the whole profit under the one aspect is, I take it, to show the exact rate of interest, (using interest as a measure of profit) realized on the transaction; and it surely is as needful to know what that rate is in contingent as in absolute reversions.

It may be urged that besides the direct interest profit, a Life Company purchasing a reversion has a right in the ordinary course of business to a further profit on the insurance involved in the transactions. To this I have no objection whatever to make so long as it is not laid down as a principle that the annuity ought to be dealt with in a different manner. Mr. Sprague's formula, however, while giving an insurance profit, does not give the same profit as would be made on a similar assurance unconnected with any reversion. The premium in the latter case would be the office single payment, while in the formula I am criticizing, it is the office annual premium multiplied by the Carlisle 6 per-cent annuity-due. Using, then, the benefit of the assurance in the ordinary sense of a like profit to that made on a similar policy unconnected with a reversion, Mr. Sprague's formula does not give the office employing it the benefit of the assurance. Neither (since ${}^{n}P_{xy}^{1}$ is not the six per-cent premium plus the loading for expenses) is the total insurance benefit excluded. So that while, on the one hand, the formula criticized will not satisfy those who in addition to the direct interest, wish to obtain also the usual profit made on assurances, it will not on the other carry out the desires of those that require all the return on the transaction to appear as interest.

I have already pointed out that the formula for a contingent reversion I am examining is inconsistent with that for an absolute reversion; and I shall now proceed to show that its various parts are inconsistent with each other. The premium employed is the office annual rate, but the annuity is on a six per-cent basis. Now, leaving aside the loading, which, as will be hereafter seen, I take no exception to, since the net premium is never founded, on this side of the Atlantic at least, on a six per-cent table, we have here a profit taken on the assurance but the annuity sold at cost price. That is, Mr. Jellicoe's formula has been changed in the one particular but not

in the other. I am at a loss to imagine why. It cannot be because the assurance might require to be shared by another office, because that is altogether a special case, and the formula is not so limited. Besides, it might hold good of the annuity also. It too might be so large that the office could not retain it all at its own risk. I cannot then see any grounds for changing the one element of Mr. Jellicoe's formula and not the other. It is not because the annuity is not bought and the insurance is, as the annuity is just as much purchased as the assurance. It is true that it is usual to write out an assurance policy and pay premiums under it from the one pocket to the other, while it is not customary so to treat the annuity. But this is a mere convenience for book-keeping and other purposes, and does not affect the matter in the least. The profit would neither be more nor less, if on the one hand an annuity bond was written out as well as a policy, or on the other, the policy were ignored exactly as is the annuity now. Like then as in the difference of treatment between absolute and other reversions, in the various parts of the latter formulæ there seem to me distinctions drawn where no differences exist.

But it may be argued on behalf of these formulæ, that the distinguishing between assurances and annuities is intended, because the expressions show specially the terms on which Reversionary Companies, the largest purchasers, may enter into such transactions, and are not meant to be primarily applicable to Insurance Societies. In this argument it is assumed that while it is quite legitimate that a Reversionary Company should undertake the risk of the annuities involved in its transactions, it should have nothing to do with the corresponding assurances. I fail to see any grounds for the distinction drawn between the two, and why it may undertake the one, and not the other. If it has not a sufficient number of the latter to form an average, it may readily happen that it has not a safe average of the former either. Why then assume it may take the risk of its own annuities but not of its own assurances? Even if it could be established that it was reasonable for a Reversionary Society to purchase its annuities from itself, but not its assurances, it by no means follows that the formulæ I take exception to are to be the market price basis. Just as Mr. Sprague objected to the individual purchaser setting the tone of the market, and transferred it to the largest buyers, it may be held that Insurance Offices ought to set the tone, as, if not the largest, yet they are dealers large enough to affect the market, and from their additional insurance profit and wider basis can easily undersell the Reversionary Societies.

For the reasons given then in the foregoing analysis, I take exception to Mr. Sprague's formulæ for contingent reversions and reversionary annuities. If the principle that interest must be the only profit is adopted in one formula, consistency requires that it should be adopted in all.

Let us now proceed to construct the expression for, say, a Contingent Reversion, on this principle, the interest realized being, say, t per-cent. First, as to the net premium. This must be taken on an interest basis of t per-cent, or else beyond the direct profit there will be an assurance profit also. Next as to the loading. Unless the transaction is so large as to require reassurance, the actual profit beyond the fee for the reversioner's medical examination, would not be affected were no policy whatever opened; and consequently, so far as the assurance is concerned, it might seem that the loading could be dispensed with. However, as has been already referred to, it is convenient for book-keeping purposes that a policy be opened precisely as if it were an ordinary case, so it may be well to add to the net premium the loading, which loading, however, should be fixed, under the conditions I am assuming, with a view to expenses merely and not to profit. Another reason for taking some loading may be found in the fact, that though it can hardly be called expense of managing the assurance, yet on realization of the reversion there may be some expense or trouble, and the premium loading may be considered as including this. This loading ought, however, to be added to a t per-cent net premium, or else a further profit than the direct interest will be taken on the transaction. No doubt the premium rate thus found would be smaller than the ordinary office premium; and, if the reversion were redeemed, it would not do to hand over the policy to the original seller of the reversion subject to such a premium only. This, however, (*which applies equally to the annuity, to which the seller has just the same claim as to the assurance*), is a mere matter of detail; and I do not assert that this premium should be the premium actually passed through the books and entered on the policy. All that is insisted on is that, in order to carry out the above principle, in the price given for the reversion, the premium should be taken at the rate realized, plus the loading for expenses.

I shall afterwards revert to the assurance premium with reference to the mortality table employed, but in the meantime I shall pass to the annuity. So far as the motives for selection are concerned, there is no difference between the sellers of life-policy reversions and the sellers of other reversions;—at least, the burden

of proof that there is, rests with those who assert the contrary. It may be assumed then that in, say, absolute reversions, the mortality experienced will be the same as that prevailing among the persons who surrender their ordinary assurance policies. We know that in policy surrenders the annuity at the date of surrender *may* be as great by the general Assured Experience (^na_x) as by the cancelled policy experience ($^na'_x$). The matter must be judged then simply on the grounds as to the sufficiency or non-sufficiency of the table taken as representative of the Assurance Experience to provide for the mortality among the lives on whom reversions depend. This is a question to be determined by *à posteriori* rather than by *à priori* considerations. In the absence of the former, however, I would be inclined rather than any assurance table to consider the experience of annuitants as the nearest representative of the mortality of the lives on whom reversions depend. In reversionary transactions there is all the selection of annuitants, with the sole exception that, unlike annuity purchasers, reversion sellers may be forced by pecuniary reasons to discount their expectations. Taking annuitants as represented by Finlaison's Government Experience (1860) we shall find that for the ages at which reversions (other than policies) are generally sold, the Carlisle Annuities are smaller than the Government Experience of females, and sometimes smaller and sometimes larger than the same experience of males. It may be well to point out here, that, so far as selection is concerned, there is a distinction between the purchase of a contingent reversion and the surrender value, if such were given, of a Survivorship Assurance, which does not exist in the corresponding sales of absolute reversions and ordinary policies. In the purchase of a contingent reversion, there is a medical examination of one of the parties, and it is equivalent to what the surrender value of a Survivorship Assurance would be if the counter life were selected by the medical officer but not the direct life.

The only other matter in this branch of the subject that requires analysis, is the effect of the introduction of varying interest and mortality rates at the same time. Taking, as before, for illustration the value of a contingent reversion, we know that its usual shape, $^nA^1_{xy}$, is perfectly homogeneous, and if, using the general expression previously given, we put it in the form

$$^nA^1_{xy}=1-\{1-v+{}^nP^1_{xy}\}\,{}^n\mathbf{a}_{xy},$$

so long as we employ the same rates throughout, we have no

difficulty. The moment however we vary the rates our troubles begin. We have already seen as to direct interest that the rate to be realized in the future must not be less than that taken in the cost of the annuity. And now coming to mortality;—If we split up the lives involved in the reversion into various sections, and endow each section with a vitality of its own, we must take care that in thus playing the part of Frankenstein we are not creating monsters to devour us. If, however, interest is to be the only profit on the reversion, since it is thus requisite that the same mortality table be employed throughout, we are spared any anxiety on this ground.

It may be well, however, before putting together the various parts of a reversion with the profit under the one aspect, to note the effect on Mr. Jellicoe's and Mr. Sprague's formulæ of the diverse interest and mortality rates they introduce into the same expression. First, as to the interest rate, which (as we have already said) must not be lower in the direct profit taken than in the annuity. Since in Mr. Jellicoe's expression the direct profit rate is higher than in the annuity, while in Mr. Sprague's it is as high, no difficulty arises on this score in either formula. Next, as to the mortality—Mr. Jellicoe, as we have seen, goes on the principle that the purchaser of the reversion is to be a person who actually buys both the annuity and the assurance. In this case, inconsistent vitalities are of no moment to the purchaser. He leaves the Assurance and Annuity Societies to settle as they please their little difficulty as to how the same man can die in two different ways. If, however, he himself is both the assurer and the granter of the annuity, it becomes quite another matter. Inconsistent vitalities in the same man might be very awkward indeed. If, then, such a purchaser employ either Mr. Jellicoe's or Mr. Sprague's formula, unless he wishes his profit to be reduced, he must not permit the table used for the assurance premium to show a lighter mortality than that employed for the annuity.

Let us now return to the fitting together of the various parts of the expression for a reversion, and we find, in accordance with the analysis we have just completed, that the value of, say, a contingent reversion, where there is to be the interest profit, the whole interest profit, and nothing but the interest profit is

$$1-\{1-\overset{t\%}{v}+\overset{t\%\,\text{Ann. Exp.}}{{}^{n}\mathrm{P}_{xy}^{\,1}}+\text{loading}\}\,.\,\overset{\text{Ann. Exp. } t\%}{{}^{n}\mathbf{a}_{xy}}$$

Let us throw this into a shape more adapted to the use of

auxiliary tables. The interest rate, t, is, say 6 per-cent. As has already been pointed out, the Annuity Experience is sometimes higher and sometimes lower than the Carlisle table. To be on the safe side, let us assume that the life aged $(x+n)$ is so much better than the Carlisle table as to reduce the premium and the Annuity rate from 6 to 5 per-cent Carlisle.

Remembering that the ordinary interest rate realized is about $4\frac{1}{2}$ per-cent, while the Annuity selling price of Mr. Jellicoe, including both profit and expenses, is Carlisle $3\frac{1}{2}$ per-cent, this one per-cent off the 6 per-cent rate of interest may be considered ample allowance for the possible selection against the office. On these assumptions, then, the working formula for a contingent reversion, where 6 per-cent interest is the only profit, would be

$$1-\{1-\overset{6\%}{v}+{}^{n}\overset{\text{Car. }5\%}{P^{\;1}_{xy}}+\text{loading for expenses}\}\,.\,{}^{n}\overset{\text{Car. }5\%}{\mathbf{a}_{xy}}$$

on corresponding assumptions, for an absolute reversion yielding 6 per-cent return, and that alone, the formula would be

$$1-\{1-\overset{6\%}{v}\}\,.\,{}^{n}\overset{\text{Car. }5\%}{\mathbf{a}_{x}}.$$

These values, from the nature of the case, cannot be considered anything else than rough approximations to the carrying out of the principle that in the purchase of reversions, the whole profit of the transaction is to appear under the one aspect of interest, that interest being here taken as 6 per-cent. In actual working it may very well happen that it is of small moment whether Mr. Jellicoe's, Mr. Sprague's or the formula here given is employed. If the value the seller of the contingent reversion gets be nearly the same (and it readily may be so) whichever of the three be used, he won't be exceedingly anxious as to whether only 5 per-cent interest has been charged him, while a profit is made on the assurance and annuity; or whether it be 6 per-cent and a profit on the assurance, but the annuity sold under (or at) cost price; or whether 6 per-cent interest be charged, and possibly no profit taken on either assurance or annuity. All that can be done by theory is to lay down the principle in the matter: the actual rates will be settled by competition. It may be objected to the formula I give, that, if the policy has to be reassured, the profit will fall below 6 per-cent, and this is no doubt the case; but the same thing holds true of the annuity. It also may be too large to be retained by an office at its own risk. So the same objection may,

so far as the annuity is concerned, be brought against Mr. Sprague's formula also. If it is wished to obtain 6 per-cent all over, while the part taken by the principal office may be valued by the preceding formula, the part requiring reassurance can be dealt with thus—

1st, if both Assurance and Annuity are to be taken out in another office, we have (Mr. Jellicoe's formula) the contingent reversion

$$1-\{1-\overset{6\%}{v}+{}^{n}\overset{\text{Office rates}}{P^{1}_{xy}}\}\,{}^{n}\overset{\text{Office rates}}{\mathbf{a}_{xy}};$$

2nd, if the assurance alone is taken out in another Society, the contingent reversion becomes

$$1-\{1-\overset{6\%}{v}+{}^{n}\overset{\text{Office rate}}{P^{1}_{xy}}\}\,{}^{n}\overset{\text{Car. }5\%}{\mathbf{a}_{xy}};$$

3rd, if Annuity alone is taken out (this case will not of course arise in consequence of the original office having more than its maximum amount at risk), the value of contingent reversion is

$$1-\{1-\overset{6\%}{v}+{}^{n}\overset{\text{Car. }5\%}{P^{1}_{xy}}+\overset{\text{Office rate}}{\text{loading}}\}\,{}^{n}\mathbf{a}_{xy}.$$

Having analyzed the formulæ for the purchase of reversions as investments, let us now proceed to the application of these expressions in calculating the surrender values of policies. Let us first take the case of the reversion purchased, say an ordinary whole life policy, being one that the purchaser is not under obligation to pay. Either Mr. Jellicoe's or Mr. Sprague's formula, or the expression

$$1-(1-\overset{t\%}{v})\,{}^{n}\overset{\substack{\text{Ann. Exp.}\\ t\%}}{\mathbf{a}_{x}} - P_{x}\,{}^{n}\overset{\substack{\text{Ann. Exp.}\\ t\%}}{\mathbf{a}_{x}}$$

may be employed. In the case of P_x, the annual burden on the absolute reversion, being a 'with profits' premium, if the seller is charged the future payments of that premium, he ought to get value for the future bonuses. In such a case, it will be requisite to assume the amount of bonus that may be reasonably expected and apply the general formula given at the beginning of this paper. Thus, if B is the reversionary annual bonus anticipated, we have previously seen that its value may be set down as

$$B\{\overline{\omega-x-n}-[(1-v)(\omega-x-n)+\mathbb{P}]\,{}^{n}\mathbf{a}_{x}\};$$

and applying the requisite rates and loadings to this expression,

the amount a purchaser can give for this future bonus, either by Mr. Jellicoe's formula or the others, will be arrived at.

We next come to the more immediate subject of the paper, viz. when the policy is purchased by the office granting it. Let us first apply Mr. Jellicoe's principle of treating each reversion as isolated. In the expression

$$1-\{1-v+\mathrm{P}_x\}\,{}^n\mathbf{a}_x,$$

we have for the annuity the office selling rate, say Carlisle $3\frac{1}{2}$ per-cent. In providing the annuity in this way, the difficulty of selection is got rid of.

In surrender values there are no further expenses or bonus, so P_x can be taken as the pure premium, which also we shall assume to be at the same interest rate as v. Column 1 in the annexed Table shows the surrender values of policies calculated in this way, the annuity being the Carlisle $3\frac{1}{2}$ per-cent, and the interest realized by the office on the transaction being taken as 4 per-cent. P_x has in each case been calculated by the Carlisle and the Institute (H^{M}) rates. It will be observed that at the younger ages it is a considerable number of years before the policies acquire any surrender value, a very serious drawback in practical working. If, in calculating the reserve, the rate of interest was not greater than $3\frac{1}{2}$ per-cent, and the office were content to realize this rate merely, Column 2 will show for the Carlisle Table the surrender values it would give. I do not furnish the Institute Values in this case, as, if the reserve rate of interest were $3\frac{1}{2}$ per-cent, the annuity at age $(x+n)$ might, by the Institute Table, be greater than the Carlisle, and the surrender value would come out larger than the reserve amount. Of course by always taking, in the Institute Surrender Values, whichever annuity was greater, this difficulty might be overcome. If, instead of assuming a profit in interest beyond the reserve rate, the office, valuing against liabilities at 3 per-cent, was willing to give in its surrenders the benefit of this rate, and to look for its profit to the Annuity, which we shall, according to the basis table, assume to be either the 3 per-cent Carlisle or Institute Annuity with 5 per-cent loading thereon. Column 3 will show the surrender values it would pay.

All these methods, however, and any founded on Mr. Sprague's formula for a contingent reversion also, will leave the office in doubt as to the exact amount of profit on the transaction. Let us,

therefore, so far as may be, attempt to bring the whole profit under the one heading of interest. The formula for this purpose is, as formerly stated, when t is the interest rate wished

$$1-\{\overline{1-\overset{t\%}{v}}+\overset{t\%}{P_x}+\text{loading}\}\,{}^n\mathbf{a}_x.$$

As a policy causes no expense after surrender, the loading in the above expression will disappear.

The question next meets us,—what mortality rate should we adopt in the annuity? We have no table showing directly the vitality of those who drop their policies; and even if we had, unless the special circumstances of the office whose experience it was, were taken into account, we might find it a rather dubious guide in the matter. In the absence of a reliable table (as I have formerly stated) the experience of annuitants seems to me the best criterion of the effect of policy lapses. It may not, and probably will not, exactly express this effect; but I think the error must be on the safe side. As a rule, an annuity is not bought unless there is a pretty shrewd guess that the life on which it is granted is a very good one. I say, 'as a rule', because, under wills, annuities are sometimes purchased though the lives are not over the average vitality; but I do not consider these cases sufficient in number to upset the conclusion that such contracts are only entered into when the life is believed to be better than ordinary. Annuitants are influenced mainly by a lively sense of favours to come. They sink money expecting to reap a profitable return for it. On the other hand, those who discontinue their policies have no such untramelled selection. Their discontinuance may be caused, not by a consciousness of superior vitality, but by an inability to pay the premium; and this last cause, it seems to me, will have an effect on their selection, sufficient to make their mortality not less than that of annuitants, despite the possibility that, in the case of the former, some vestiges of the medical selection may yet remain to diminish their death rate. Therefore, though it is impossible to speak with mathematical certainty, it seems to me, only very strong *à posteriori* evidence would justify us in considering the mortality among annuitants greater than among policy discontinuers. I shall therefore assume that the selection in the surrender of policies will be fully met by adopting an Annuity Experience Table; and for practical application I shall, as in ordinary reversions, take Finlaison's (1860) Government Annuitants Experience,

"Males of the Combined Observations." Comparing then say, the "Institute Healthy Males," with the above, we shall find that at the earlier ages the Medical Selection table shows the larger annuity; but at age 50 (not the one proverbially set down as that at which every man is either a fool or a physician) and upwards, self-selection beats the doctor. This result is not surprising when we call to mind the greater influx at the younger ages of new lives in the assurance table. The fact that it is so, however, compels us, as our surrender values must not exceed the reserve amounts, to take, when the annuity table is the lower, the assurance rates.

Passing to another element of the expression

$$1-\{1-v+\mathrm{P}_x\}\,{}^n\mathbf{a}_x$$

we have formerly seen that a policy by annual premiums may be looked on as an absolute reversion burdened with an annuity of P_x. P_x is quite independent of ${}^na_x$, and, though to bring the profit as much as possible under the one aspect I have in it employed t per-cent interest, yet its mortality is that of the table used at the time the policy was first issued.

The only other element in the expression is the interest rate to be taken for t. This may be anything we please, subject merely to the limitation that t does not fall under the rate employed for reserve purposes. Looking at the purchase by a solvent office of its own policies purely as an investment, the investment is in safety far superior to the best possible reversion of another kind. In the redemption by the office of its own obligations, there is no future expense, no danger of the security being insufficient, and no chance of dispute afterwards as to the adequacy of the consideration money. If, then, in other reversions 6 per-cent interest be considered a satisfactory return, less than that rate may suffice for policy surrenders. In fact, since the purchase of (or loan on) its policies, by a solvent office is an investment better secured and less troublesome than any other whatever, whether it be government securities, mortgages on land, or railway debentures, consistency demands that the interest rate in policy surrenders should not exceed the minimum investing rate for securities not readily realized. And what is called for by consistency seems to me also the most politic course in the interests of the office. This, however, is a matter of individual opinion and not one that can be settled *à priori* by theoretical considerations.

We have now considered the component elements of the formula, and all that remains is to furnish a few instances of its working. In applying the foregoing reasoning it will be necessary to consider the reserve rates employed by the various offices. The following illustrations are as before taken from the Institute H^M and the Carlisle Tables at various rates of interest. The Institute values, will, from the great similarity between this table and the Old 17 Offices Experience, indicate the method of treatment for the latter also. It has already been stated that when the Medical-selection Annuity is greater at any age than the Self-selection Annuity, the former should be used in our calculations. In practice, this break of gauge, as I may call it, will be not only inconvenient, but will also in a few policies of short duration lead to somewhat anomalous results. It will therefore be expedient, if we can do so safely, to employ the one table throughout. Whether this may be done or not, can only be settled by comparing the mortality table used for estimating the liabilities with the Annuity Experience table proposed to be the basis of the surrender values. Thus, if the liabilities be calculated by the H^M or the 17 Offices Experience, at ages 50 and 48 respectively, the annuities by these tables begin to fall under the Government Annuities. In practice, the bulk of the surrenders occur prior to these ages. An office then which values its liabilities by either of these Assured Experience rates, can afford to employ the same table of mortality for its surrenders; the few cases in which the Government Annuity would give a greater value being compensated by the many in which it would give a less. Or, taking another table. The Carlisle Annuity dips under Finlaison's Government Annuity at age 57, and after that age has a somewhat chequered. career, being at one time higher and at another lower than the Government rate. If, then, it is safe for an office valuing its liabilities by the Institute table to employ that table for surrender values, *à fortiori* a society employing the Carlisle for reserve purposes can use it also for surrenders. And so with the other tables in recognized use for the valuation of Assurances, either as they stand, or by taking a higher interest rate than that hoped to be realized can the selection difficulty in surrender values be got over.

Acting on this reasoning, Column 6 shows the surrender values of policies by the Institute and Carlisle Tables, allowing the office nominally 5 per-cent interest on the purchase. I say "nominally" because, as the bulk of surrenders take place before ages 50 and

57 respectively, the actual interest realized will be more than this. As 6 per-cent is the market rate for ordinary reversions, I furnish in Column 8, for the purpose of comparison, the values by the same tables at this rate also. For the reasons given previously, however, I consider an office in purchasing its own policies can afford to take a somewhat lower rate than this.

I have said that P_x the premium is independent of the mortality table used in the present annuity at age $(x+n)$, and so far as mortality was concerned I have so treated it, but in order to bring the profit as much as possible under the one aspect, the interest in P_x has been taken as the realized rate. Employing however for P_x, the reserve rate, say 3 per-cent, Column 7, will show the surrender values of policies by the Carlisle and Institute (H^M) tables when (the present annuity being by these same tables) beyond 5 per-cent a further profit is charged by taking P_x at 3 per-cent.

Column 5 shows the values by the same tables where an arbitrary deduction of 40 per-cent is made from the pure 3 per-cent values. As I stated at the beginning of this paper, I believe this is the method of calculating surrender values in common use at present; and I think the deduction made in the illustrations is about the average amount taken off from a 3 per-cent value. A comparison with Column 8 will show that in policies of long duration, the arbitrary deduction method of calculating surrenders gives results smaller even than the 6 per-cent interest values. And thus an office calculating surrender values by Column 5, while willing to purchase less fully secured and more troublesome reversions at a 6 per-cent rate of interest, declines to buy its own better secured and less troublesome policy-reversions at that same price.

We have now examined the problems I proposed to take up this evening. These were, *First*—the principle to adopt in calculating the surrender value of a policy, and *Second*—how to apply this principle in practice. The former of these I have attempted to solve by making the subject of this paper simply a case of the wider formula given by me; showing that, whether it be a question of premium to charge for a benefit, or whether it be the investment price of a reversion, or whether it be the surrender value to be allowed for a policy, the formula and principle were always the same, and the only point requiring study was the rates to be employed in the various parts of that formula. The examination of these rates formed the second problem. In it the main difficulty

was selection. This I considered was fully met by the use of Annuitants Experience; and for that Experience, I preferred, in the present state of our statistics, the Government Annuitants'. In practice, this principle of calculating surrenders will generally come to a pure-premium valuation by the table of mortality employed in estimating the reserve and by such rates of interest as may be wished to be realized on the transaction. And the result, as compared with the arbitrary deduction method, will be that the surrender values of policies of long duration will by the mode of calculation here advocated be greatly increased.

Finlaison's Government Annuitants (1860) (*Males*) *Combined Observations*—5 *per-cent.*

Age.	Annuity.	log Annuity-due.	Age.	Annuity.	log Annuity-due.	Age.	Annuity.	log Annuity-due.
15	15·459	1·21640	39	13·390	1·15806	63	8·193	0·96346
16	15·351	1·21354	40	13·250	1·15381	64	7·911	0·94993
17	15·252	1·21091	41	13·103	1·14931	65	7·629	0·93598
18	15·166	1·20860	42	12·947	1·14448	66	7·350	0·92170
19	15·093	1·20664	43	12·781	1·13928	67	7·075	0·90715
20	15·034	1·20504	44	12·605	1·13370	68	6·805	0·89237
21	14·987	1·20377	45	12·419	1·12772	69	6·540	0·87739
22	14·948	1·20271	46	12·224	1·12136	70	6·282	0·86223
23	14·912	1·20172	47	12·020	1·11461	71	6·028	0·84681
24	14·875	1·20071	48	11·810	1·10755	72	5·777	0·83101
25	14·831	1·19951	49	11·595	1·10020	73	5·526	0·81466
26	14·778	1·19805	50	11·377	1·09262	74	5·275	0·79763
27	14·714	1·19629	51	11·157	1·08483	75	5·022	0·77977
28	14·639	1·19421	52	10·936	1·07686	76	4·770	0·76115
29	14·554	1·19184	53	10·714	1·06871	77	4·521	0·74198
30	14·459	1·18918	54	10·490	1·06032	78	4·279	0·72252
31	14·356	1·18628	55	10·262	1·05162	79	4·047	0·70305
32	14·247	1·18318	56	10·029	1·04254	80	3·829	0·68385
33	14·134	1·17995	57	9·789	1·03298	81	3·624	0·66500
34	14·017	1·17658	58	9·542	1·02292	82	3·431	0·64645
35	13·898	1·17313	59	9·286	1·01225	83	3·247	0·62806
36	13·776	1·16950	60	9·022	1·00095	84	3·070	0·60959
37	13·651	1·16587	61	8·751	0·98904	85	2·897	0·59073
38	13·523	1·16206	62	8·474	0·97652	86	2·725	0·57115

Surrender Value of ordinary Policies for £100 when the profit appears chiefly under the one aspect of interest.

Entry Age (x)	Duration of Policy (n years.)	$_{n}\vert V_x = 1 - \{1 - v + P_x\}\,{}^{n}a_x$ Rates the same throughout. (Column 4.)		Column 4 − 40 % of value of policy. (Column 5.)		$_{n}\vert V_x = 1 - \{1 - v + P_x\}\,{}^{n}a_x$ Rates the same throughout. (Column 6.)		$_{n}\vert V_x = 1 - \{1 - v + \overset{3}{P_x}\}\,{}^{n}a_x$ Rates the same throughout, except P_x, which is by same mortality table, with interest at 3 %. (Column 7.)		$_{n}\vert V_x = 1 - \{1 - v + P_x\}\,{}^{n}a_x$ Rates the same throughout. (Column 8.)	
		Car. 3 %.	Inst. H^M 3 %.	Car. 3 %.	Inst. H^M 3 %.	Car. 5 %.	Inst. H^M 5 %.	Car. 5 %.	Inst. H^M 5 %.	Car. 6 %.	Inst. H^M 6 %.
25	1	1·029	1·015	0·617	0·609	0·714	0·677	—	—	0·606	0·558
	2	2·094	2·069	1·256	1·241	1·463	1·386	—	—	1·247	1·149
	3	3·157	3·134	1·894	1·880	2·218	2·116	—	—	1·893	1·762
	4	4·173	4·214	2·504	2·528	2·922	2·866	—	—	2·490	2·389
	5	5·119	5·313	3·071	3·188	3·562	3·626	—	—	3·018	3·029
	10	10·305	11·121	6·183	6·673	7·221	7·813	8 + 2·206	8 + 2·425	6·110	6·597
	15	16·262	17·524	9·757	10·514	11·743	12·707	6·973	7·605	10·061	10·875
	20	22·169	24·703	13·301	14·822	16·233	18·591	11·769	13·833	14·031	16·180
	25	29·371	32·407	17·623	19·444	22·350	25·276	18·154	20·908	19·544	22·377
	30	38·111	40·584	22·867	24·350	30·401	32·808	26·640	28·880	27·204	29·536
	40	54·226	57·265	32·536	34·359	46·240	49·428	43·334	46·472	42·716	45·920
	50	69·944	72·496	41·966	43·498	63·265	66·078	61·280	64·095	60·155	63·046
35	1	1·286	1·395	0·772	0·837	0·924	1·001	—	—	0·791	0·853
	2	2·598	2·808	1·559	1·685	1·877	2·035	—	—	1·610	1·744
	3	3·932	4·243	2·359	2·546	2·857	3·090	—	—	2·458	2·656
	4	5·293	5·709	3·176	3·425	3·868	4·184	—	—	3·337	3·604
	5	6·642	7·204	3·985	4·322	4·874	5·309	—	—	4·208	4·580
	10	13·228	15·281	7·937	9·169	9·779	11·692	6 + 4·536	6 + 6·170	8·437	10·259
	15	21·257	23·950	12·754	14·370	16·307	18·943	11·444	13·875	14·308	16·894
	20	31·001	33·150	18·601	19·890	24·985	27·113	20·626	22·556	22·466	24·559
	25	40·868	42·636	24·521	25·582	34·289	36·007	30·470	32·006	31·451	33·116
	30	48·967	51·918	29·380	31·151	42·056	45·142	386·89	41·712	38·989	42·100
	40	66·491	69·054	39·895	41·432	60·406	63·203	58·105	60·902	57·562	60·436

45	1	1·651	2·013	0·991	1·208	1·231	1·578	—	—	1·065	1·415
	2	3·377	4·042	2·026	2·425	2·541	3·186	—	—	2·209	2·853
	3	5·202	6·080	3·121	3·648	3·960	4·823	—	—	3·459	4·331
	4	7·172	8·148	4·303	4·889	5·535	6·500	—	0·652	4·872	5·841
	5	9·253	10·232	5·552	6·139	7·236	8·211	1·506	2·470	6·412	7·394
	10	20·483	21·092	12·290	12·655	16·854	17·463	11·718	12·300	15·322	15·934
	15	31·854	32·289	19·112	19·373	27·167	27·535	22·668	23·002	25·135	25·470
	20	41·187	43·246	24·712	25·948	35·776	37·879	31·809	33·993	33·367	35·481
	25	51·827	53·860	31·096	32·316	46·247	48·390	42·927	45·162	43·696	45·880
	30	61·383	63·472	36·830	38·083	56·115	58·331	53·404	55·725	53·652	55·913
55	1	2·942	2·826	1·765	1·696	2·506	2·371	—	—	2·321	2·184
	2	5·926	5·659	3·556	3·395	5·082	4·786	—	—	4·725	4·415
	3	8·881	8·500	5·329	5·100	7·660	7·229	2·481	1·999	7·140	6·685
	4	11·690	11·348	7·014	6·809	10·121	9·708	5·080	4·617	9·448	9·000
	5	14·300	14·190	8·580	8·514	12·403	12·203	7·490	7·253	11·588	11·344
	10	26·038	28·075	15·623	16·845	22·757	24·736	18·424	20·492	21·310	23·251
	15	39·418	41·526	23·651	24·916	35·351	37·471	31·725	33·946	33·508	35·622
	20	51·435	53·709	30·861	32·225	47·219	49·515	44·258	46·669	45·266	47·557
65	1	3·419	3·757	2·051	2·254	2·994	3·330	—	—	2·808	3·143
	2	6·951	7·496	4·171	4·498	6·134	6·681	1·823	2·499	5·772	6·317
	3	10·577	11·229	6·346	6·737	9·398	10·059	5·237	6·028	8·872	8·531
	4	14·300	14·976	8·580	8·986	12·795	13·483	8·791	9·606	12·124	12·811
	5	18·091	18·702	10·855	11·221	16·305	16·921	12·461	13·198	15·501	16·118
	10	34·339	35·639	20·603	21·383	31·669	32·920	28·531	29·917	30·443	31·669
	15	45·906	49·644	27·544	29·786	42·790	46·601	40·163	44·208	41·336	45·170
75	1	3·608	4·608	2·165	2·765	3·293	4·261	0·342	1·256	3·152	4·104
	2	6·954	9·067	4·172	5·440	6·345	8·410	3·488	5·535	6·073	8·110
	3	10·348	13·402	6·209	8·041	9·468	12·469	6·707	9·722	9·069	12·043
	4	14·130	17·649	8·478	10·589	13·008	16·480	10·354	13·859	12·494	15·944
	5	17·617	21·760	10·570	13·056	16·276	20·380	13·722	17·893	15·661	19·758
	10	35·054	38·318	21·032	22·991	33·070	36·398	31·028	34·299	32·141	35·350

Surrender Values of ordinary Policies for £100 by modifications of Mr. Jellicoe's formula

Entry Age (x).	Duration of Policy (n).	$_{n\|}V_x = 1-\{1-v+P_x\}\,\overset{\text{Car. }3\frac12}{{}^n a_x} = 1-\frac{\overset{\text{Car. }3\frac12}{{}^n a_x}}{a_x}$ (Column 1.) Reserve on Car. Basis. P_x=Car. 4 % prem.	Reserve on Inst. (H^M) Basis. P_x=Inst. (H^M) 4 % prem.	$_{n\|}V_x = 1-\frac{{}^n a_x \text{ Car. }3\frac12}{a_x \text{ Cr. }3\frac12}$ (Column 2.)	$_{n\|}V_x = 1-\{1-\overset{3}{v}+P_x\}(1+\frac{21}{20}\,{}^n a_x)$ (Column 3.) Reserve on Car. Basis. P_x and a_x by Car. 3%.	Reserve on Inst. (H^M) Basis. P_x and a_x by Inst. 3%.
25	1	—	—	0·937	—	—
	2	—	—	1·907	—	—
	3	—	—	2·880	—	—
	4	—	—	3·803	—	—
	5	—	—	4·661	·606	·807
	10	8 + 2·559	6 + 4·177	9·408	6·050	6·906
	15	8·542	10·061	14·970	12·309	13·627
	20	14·505	15·924	20·514	18·509	21·163
	25	21·943	23·239	27·430	26·072	29·254
	30	31·179	32·322	36·016	35·246	37·839
	40	48·498	49·353	52·118	52·168	55·354
	50	65·825	66·392	68·227	68·672	71·347
35	1	—	—	1·183	—	—
	2	—	—	2·391	—	—
	3	—	—	3·626	—	—
	4	—	—	4·887	·826	1·252
	5	—	·835	6·140	2·231	2·819
	10	6 + 6·455	7·300	12·260	9·147	11·299
	15	14·594	15·365	19·893	17·579	20·402
	20	24·699	25·379	29·371	27·808	30·061
	25	35·081	35·668	39·109	38·170	40·022
	30	43·649	44·158	47·146	46·673	49·768
	40	62·607	62·945	64·927	65·073	67·762
45	1	—	—	1·535	—	—
	2	—	—	3·145	—	—
	3	—	—	4·861	·757	1·687
	4	1·560	·207	6·722	2·828	3·856
	5	3·648	2·324	8·700	5·014	6·045
	10	15·049	13·881	19·503	16·802	17·446
	15	26·761	25·755	30·602	28·744	29·204
	20	36·427	35·553	39·761	38·543	40·707
	25	47·629	46·909	50·375	49·716	51·854
	30	57·815	57·235	60·027	59·749	61·947
55	1	—	—	2·824	—	—
	2	1·615	—	5·696	1·594	1·320
	3	4·591	2·553	8·549	4·696	4·307
	4	7·426	5·448	11·266	7·649	7·300
	5	10·056	8·135	13·788	10·389	10·280
	10	21·927	20·259	25·166	22·713	24·858
	15	35·684	34·310	38·352	36·764	38·985
	20	48·193	47·086	50·342	49·380	51·775
65	1	0·232	—	3·304	—	—
	2	3·772	—	6·735	2·804	3·402
	3	7·415	2·853	10·265	6·621	7·323
	4	11·168	6·791	13·903	10·519	11·256
	5	15·004	10·815	17·620	14·501	15·168
	10	31·535	28·161	33·642	31·559	32·951
	15	43·357	40·565	45·100	43·706	47·658
75	1	1·465	—	3·526	—	·666
	2	4·802	—	6·793	3·070	5·344
	3	8·196	1·798	10·116	6·634	9·895
	4	11·996	5·863	13·837	8·524	14·355
	5	15·499	9·610	17·267	12·347	18·674
	10	33·144	28·485	34·542	32·576	36·059

The following account of the discussion which followed the reading of the paper is abridged from the *Insurance Record.*

Mr. SPRAGUE congratulated the Institute upon the very interesting paper they had listened to. Mr. Macfadyen had shown great industry and originality of thought, and had given them all much to think about. He proceeded:—The subject of the paper is an eminently practical one. The question of surrender values occurs to all of us, we may say, every day of our lives. When we have to value a policy for surrender, I suppose that generally we have some fixed rule of doing it; we turn to our tables and quote our values, without considering much whether the value can be defended on strict theoretical principles. Then Mr. Macfadyen comes and criticizes our practice; and we must admit that it is good for us to be habitually brought back to first principles, to be obliged to consider whether the processes we adopt are thoroughly defensible, and whether we can gain any suggestions for their improvement. Now, while I am strongly of opinion that in questions of this kind theory is very useful, I hold that practice is still more useful; that the practical considerations which arise, for instance, from competition, and from the expectations of the public, are really much more important, if the two must be put in competition, than the requirements of strict theory. Therefore the paper, being exclusively a theoretical one, does not, in my opinion, assist us much in the practical treatment of the subject. As I understand, the general result Mr. Macfadyen arrives at is that we should give the assured the net 5 per-cent values of their policies; but I must say that when we have it stated in that definite way, it does appear to me that the conclusion is a very lame and impotent one to follow on such an elaborate investigation, with so many formulas as we have had placed before us this evening. Then, what is the reason for fixing on this 5 per-cent net value as the proper one to give the assured? The principal reason, as I understand it, adduced for valuing policies in this way is that by this means an office purchases its own policies on such terms as to realize 5 per-cent. Looking a little further into the matter, the question suggests itself, 5 per-cent on what? I must say that the paper does not give me any assistance towards answering that question. Is it 5 per-cent on the reserve value of the policy? It cannot be that, because the value quoted is quite independent of the reserve. Mr. Macfadyen leads us to conclude that whether the office values its liabilities at 3, 3½, or 4 per-cent it makes no difference as to the terms on which the office is to purchase its own policies. It cannot, again, mean that the office is to make 5 per-cent upon the surrender value paid, and no further profit; for the profit realized must depend on the sum reserved as the value of the policy. It seems to me that the omission to consider the amount of the reserve thoroughly vitiates the whole of his reasoning upon this subject. He has nowhere in his paper referred to the sum the office reserves for its policies; and that ought to be put in the closest possible connection with the surrender value of the policy. You must consider what you have reserved as the value of the policy before you can consider what you shall give for the surrender of it; bearing in mind that the profit on the surrender of a

policy will not be exactly the difference between the reserved value and the surrender value, because, as is admitted, the lives that surrender their policies are on the average better lives than those that keep them in force; so that a certain deduction must be made from that apparent profit in order to get the real profit on the transaction. Then, Mr. Macfadyen having laid down the principle that we are to value at 5 per-cent, how does he proceed? When we are asked to advise an intending purchaser of a policy in another office as to the market value of the policy, the first thing we do is to enquire what is the office premium. That is a consideration which he throws entirely on one side, as quite immaterial. Having thus got rid of the question of loading, he proceeds to say it is usual to value net premiums, and he will therefore assume P to be the net premium; and then he says he will assume P to be calculated at the same rate of interest as v; that is, he takes a 5 per-cent net premium. These are two assumptions which he has said nothing whatever in defence of. While a policyholder is assured, you charge him, we will say, a 3 per-cent premium; suppose you discard the loading and value only the net premium, you charge him the 3 per-cent net premium. Now, when he wants to surrender, is it right to the remaining members that you should charge him a 5 per-cent net premium? It appears to me a gross injustice towards them.

Then Mr. Macfadyen's reasoning as to the effect of selection is not at all satisfactory to my mind. I will not examine it in any detail, but just trace out the principal steps of it. He first of all says there is no doubt it has been proved that the mortality amongst those who surrender their policies is less than amongst those who keep them in force; and that therefore you should value a policy for surrender by a table showing a less mortality than that amongst ordinary assured lives. He then assumes that the Government annuitant tables will fairly represent the mortality among those who surrender; but on comparing these tables with the assured-life tables, he finds that for a great part of the table the Government annuities are less than the annuities obtained from the experience of assured lives, and thereupon he goes back again to the ordinary table of assured lives, and values policies for surrender by the *same* table as is used in calculating the premiums and reserves. That appears to me rather like reasoning in a circle.

Mr. Macfadyen tells us that he only knows of two reasons why, in valuing a policy for surrender, a deduction should be made from the amount reserved for the liability,—the one has reference to the rate of interest, and the other to the different mortality amongst those who surrender. But surely there is another very important reason. Take the simplest case, that of a man joining a mutual insurance company,—and this question of the surrender of policies is much more clearly considered when you fix your attention on a mutual company than when you take the case of a proprietary company, altho' that is a practical distinction which Mr. Macfadyen has not touched upon in his paper. If a man has joined a mutual insurance society, he has virtually agreed to make a certain contribution towards the expenses of carrying on the business; and if he withdraws, his fellow members are deprived of his contributions towards

the expenses, and the percentage of expenses charged to all the remaining members is increased. Therefore, I cannot help thinking, it would be very just and fair that something should be deducted from the value of his policy on account of the increased expenses to the remaining members occasioned by his withdrawal.

Then, again, in the particular passages where Mr. Macfadyen has spoken of the anomalies and inconsistencies which arise from the existing practice, it would have been much more satisfactory if he had expanded those points. He speaks of an actuary being called upon one day to value a policy for surrender, and the next day to value it for a purchaser. If he had considered that question thoro'ly, and shown the different values brought out by the two methods of valuation now in use, I think it would have been an extremely interesting and valuable inquiry. He points out very forcibly that when an office purchases its own policies, it virtually becomes a dealer in annuities. That is perfectly true in one sense, although he has overlooked the practical distinction between that case and the one where the company actually grants annuities. When an office receives money from the public and grants annuities, it has not only to meet the expense of keeping an office and a sufficient staff for the conduct of its business, but it has further to find investments constantly for its money, and to run the risk of the money being unemployed for a time, and the rate of interest falling below that at which the annuity has been calculated; but when a policy is surrendered, there is no further expense, and no trouble or uncertainty about the investment of the sum paid, so that the practical questions are very different in the two cases; and that is a circumstance which cannot be overlooked. Mr. Macfadyen talks about the anomaly of an office granting annuities on easier terms than it will grant assurances. I cannot say I think that an anomaly. But he looks at these questions from quite a different point. He seriously recommends us, when we buy a contingent reversion, to charge a 6 per-cent net premium with only sufficient loading to cover the expense of the medical examination and the realization of the reversion; and holding views of that sort, of course we can understand how, in his opinion, it is an anomaly. But I think the majority of us will not quite agree with him in so thinking. The difference between the management of an annuity business and an assurance business is so great that I do not think it is a matter of wonder at all that the rates charged should be differently calculated. An assurance business has to be conducted at great expense. We are obliged to have agents travelling about, or to adopt other means to push the business; but an annuity business is quite different. People find you out without your going to any trouble, and you need not even advertize to get business. Therefore I do not see that that is an anomaly.

Then there is another important point that I commend to Mr. Macfadyen's consideration,—and that is, the risk attaching to the grant of assurances is very different from that attaching to the grant of annuities. If you grant an assurance for £5000, you may be called upon the next day to pay it, and therefore you must have some substantial consideration for that risk. But if you receive £5000 for the purchase of an annuity, the only risk you run is that the annuitant

will live to an extreme old age, which is a risk that will only develop itself in process of years. Therefore the consideration for running that risk is very much less than the risk attaching to the grant of an assurance. The question of the risk attaching to different life contingencies is one which has not met the attention that its importance deserves,—at all events in England. It has been much more attended to in Germany; and to those gentlemen who, like Mr. Macfadyen, are re-examining the basis of our theory, I think there cannot be a more deserving subject for their attention.

I should now like to call your attention to a part of the paper where it appears to me that the reasoning is of an extremely involved, not to say obscure, character. It is very difficult to follow, and therefore I will not take you through it step by step, but just point out one or two features. Mr. Macfadyen says, "It follows that the selection can be met in precisely the same manner as the purchaser of a reversion may protect himself against unusual longevity in the *cestui qui vie*." I cannot at all agree with that conclusion. I think the two things have nothing whatever to do with each other; and, holding that opinion, I am not at all surprized to find that his adoption of the supposition lands him in the extraordinary absurdity that you may apparently give for the surrender value a larger sum than you have really reserved to meet the liability under the policy. He is at great pains to explain how that absurdity has arisen; but he does not trace it to its real cause, and he fails to notice that one cause which contributes towards it is the fact that P_x is assumed to be the net premium instead of the office premium.

I will now pass on to the point in which I am more personally interested—to the attack Mr. Macfadyen has made upon my formulas for contingent reversions and also for reversionary life interests. He has attacked them both, but most of his observations relate to the formula for contingent reversions. I cannot understand how he has managed to get this question into the paper at all: it does not seem to me to have any close connection with the general subject of the paper, which is that of the purchase of a policy in which only one life is involved. I have however no objection whatever to make to Mr. Macfadyen's criticisms on that account. Of course, anything we publish in our *Journal* is open to fair criticism, and I should be the last person to object. On the contrary, I am very glad to have the matter discussed. He says that, "the formula for contingent reversions, since it contains a profit distinct from the 6 per-cent interest, is at variance, as I understand it is intended to be, with the principle laid down for an absolute reversion. I can see no reason if, in the case of the latter, it is much more satisfactory that the whole profit of the transaction should be included under one rather than under several heads, why in the contingent reversion the whole profit should not also be included under the one heading of interest." He has overlooked the general idea that runs through my paper on the Valuation of Reversionary Life Interests. I commence first with absolute reversions; I say the reversionary societies are the largest purchasers of absolute reversions, and the prices they give will therefore principally determine the market price,—and it is to be observed that my paper throughout is intended to deal with what I consider to

be a fair market price, such as I could recommend a purchaser, whether an individual or a company, to give for a reversion. Well, the reversionary society requiring, as I believe they do, to pay about 6 per-cent to their shareholders, must purchase their reversions to pay at least 6 per-cent. Then I say it is better that the reversions should be so valued as to show that the companies do get 6 per-cent; and if you value them by Mr. Jellicoe's formula, assuming 5 per-cent in one part of the formula, and $3\frac{1}{2}$ per-cent in the other, you do not get your 6 per-cent in all cases—the young lives give you more than 6 per-cent and the old lives less than 6 per-cent, and therefore the use of that formula for old lives is extremely misleading for reverversionary societies. But when you come to contingent reversions, it is a different matter. The principal business of the reversionary societies is to buy absolute reversions, and it is comparatively few contingent reversions they buy. When a reversionary society buys contingent reversions, it is entitled to say—"We cannot afford to run the risk of losing our money by the death of the reversioner, and therefore we must effect an insurance." It therefore goes to an insurance office, and has to pay the office premium, and it pays the annual and not the single office premium; and of course the premium it actually pays should enter into the formula. Now, I say further that if the reversionary society, for good reasons, chooses to retain part of the risk, it is entitled to the same consideration for the risk as it would have had to pay to an insurance company, so that there is no difference in the formula. According to general practice, the reversionary societies do retain a portion of the risk in purchasing contingent reversions, and so get a part of the insurance profit. In that way they get their profits increased somewhat beyond the 6 per-cent which they calculate on, and that increase will of course help to pay the expenses of management.

Mr. Macfadyen cannot understand why the reversionary society should retain the annuity risk itself, and should not retain the assurance risk. I have already pointed out one reason for that, which is that the insurance risk is of a different kind to the annuity risk,—much more dangerous, and such as cannot be safely run. Another equally important reason is that the reversionary society makes it its business to buy reversions and to run the attaching annuity risk; and when once its business is established, it will have sufficiently numerous transactions to be able to run those annuity risks; whereas it is unlikely it will ever have sufficient transactions to run the full assurance risk as well. Then there is another important point in regard to which I certainly join issue with him. He says, that just as I transfer the control of the market from private purchasers to the largest class of purchasers, the reversionary societies, it is only right, as insurance companies go into the business, that the control of the market should be transferred to them. No doubt that will be so in time. If insurance companies make up their minds to buy reversions and bid against the reversionary societies, they can drive the latter out of the market. They can afford, if they choose, to buy them to pay only 5 per-cent. We have not, however, come to that yet, and I hope we shall not. In my former paper, I considered that the insurance companies would pay the same price as the reversionary

societies, and the only case in which I thought the former might properly undersell the latter was in the purchase of reversionary life interests; and there I thought the business was so extremely advantageous to insurance companies, that they could afford to do it on lower terms than the reversionary societies. Instead of charging 6 per-cent all thro', they might be satisfied with 6 per-cent before the reversion fell into possession, and with 5 per-cent afterwards. The same principle, no doubt, applies when competition becomes more active. If ever there should be a difficulty in finding good mortgages at 4¼ or 4½ per-cent, then, perhaps, what Mr. Macfadyen points to will take effect; but I hope it will be a long time first.

These points fully justify me, I think, in what I have said—that the author, as a theoretical man, has given undue weight to points that practical men think are undeserving of weight, and has overlooked entirely some important considerations to which practical men are obliged to pay attention; and, therefore, however instructive and valuable this paper may be, I cannot say that it has contributed anything to the practical part of the subject.

The PRESIDENT—I will now, gentlemen, ask you to pass a vote of thanks to Mr. Macfadyen for the paper he has read to us this evening, and with which he has taken so much pains. It is quite true, as Mr. Sprague says, that he has treated a practical subject in a theoretical manner, but perhaps Mr. Macfadyen means to follow it up by putting his theory into practice; and if so, I am sure we shall all be very glad to listen to him. It is difficult to lay down any rules that will satisfy the public, because we all know that many persons would like to have, and almost expect to have, their premiums back with compound interest. If Mr. Macfadyen should favour us with a continuation of this paper, one might suggest to him to consider whether it would not be feasible to come to some agreement with the policyholders as to the scale of return that they should have. If this could be determined, it would do away with a great many unpleasant misunderstandings, and then the matter would become a bargain. I am afraid it is hardly possible to do it with the variety of assurances we have, but it might be done in some cases.

Mr. MACFADYEN, in reply, said—Mr. Sprague left till the latter part of his remarks his answer to my attack on his formulas for reversions. However, I prefer to have it the head and front, and shall therefore deal with it first. He thinks the point has very little connection with the subject of my paper, and seems to consider it has been dragged in unnecessarily. I don't think so. The surrender value paid by an office has been treated by me precisely as a purchase by the office of a reversion as an investment, and, therefore, it was requisite to consider the various formulas in existence for the price to be paid for a reversion. I found formulas, by Mr. Sprague, professing to give this price, so it was of course directly in my road to analyze them. Neither can I understand why he considers I have overlooked the principle on which his formulas are based. The principle he now tells us they are founded on is precisely the one which, in the latter paragraphs of my analysis, I have examined and found wanting. Nor do I yet consider he has disposed of the charges of inconsistency I have brought against them. If, as I said in my paper, a reversionary

society which, say, keeps its annuities but not its assurances, ought to have the whole profit under one aspect, why should not an assurance society retaining both classes of business have it also? Mr. Sprague's answer is that the largest bidders are the reversionary societies, and therefore the price they can afford to give ought to set the tone of the market. Even if this were the case—and I certainly do not admit that it must be so—all that this would do would be to give a reason why the actual *price* paid by assurance offices should not be far from that given by reversionary societies; but it does not explain why one formula is objectionable to a reversionary society because there are two profits in it, while assurance offices may, in another formula, have two profits, but not three, or one. Why should assurance offices employ a formula open to the very same objection that Mr. Sprague brought against Mr. Jellicoe's formula? The very reasons given by Mr. Sprague for changing Mr. Jellicoe's formulas to suit the assumed requirements of a reversionary society, render the changed formulas unsuitable for the actual requirements of an assurance office. I say "assumed" advisedly, for even in a reversionary society there is no reason why annuities may be sold while assurances may not be. It is true that in assurances the maximum risk is at the beginning, and in annuities it is at the end. This simply means, however, that a larger average of the former is necessary, not that they cannot be meddled with at all, and a general formula intended to replace one on an intelligible principle, ought not, I think, to asume that in the one case it will have a sufficient average, but in the other not. It may also be pointed out that an annuity connected with a reversion involves more outlay at first than an ordinary annuity, as till the reversion falls in, the former is all outlay, while in the latter there is a sum in hand. I would take no exception whatever to Mr. Sprague's formulas if they had been intended only to be applicable in special cases, but they claim general acceptance, and are given to supersede those founded by Mr. Jellicoe on a general principle.

Next,—as to his general criticism of my paper. He considers I have given no reason on the 5 per-cent return surrender values for taking the premium at 5 per-cent. The reason was given in my criticism of his formulas. If I had, as he has there done, taken an office premium, I should have violated the principle he insists on in absolute reversions, but ignores in contingent reversions. And after my analysis of his formula on this very point, why he should characterize the rate chosen by me as a gross injustice that no reason was given in support of, I cannot imagine.

It has been urged also by him, as an objection to the principle of calculating surrender values advocated by me, that excepting that the former must not exceed the latter, there is no relation between the surrender and the reserve value. This is quite true, and it *is* anomalous that one office valuing its liabilities at 3 and another valuing them at 4, should pay the same surrender value. The anomaly, however, is not mine. It arises from the offices charging different prices for the same benefit. I have considered surrender value paid as simply the investment by an office at investing prices, on a reversion. The policyholder, whether the

office values at 3 or 4 per-cent, has the same thing to sell—a sum payable at his death, and it is no part whatever of the bargain that the office purchasing has to pay the reversion, and that it has made greater or less provision therefor. The policyholder has no lien on a certain sum called the value of his policy. He has simply a reversion to sell. Looking at the question in this light it was no part of my purpose to show any other relation than the one I have named between the surrender and reserve values; and the only profit I had to do with, was the direct profit realized on the investment by the office. If, tho' in principle it is uncalled for, in practice a relation beyond the absolutely-necessary one be wished to be made between the surrender and reserve policy values, the method I advocate readily assists us. Thus if the latter be at 3 per-cent, the former may be say 5 per-cent. If the latter be 4 per-cent then the former can be say 6 per-cent.

Again, as to the argument that some deduction should be made from the price paid for the investment on account of the increased ratio of expenses to remaining members, it seems to me to prove too much. Withdrawals also may be said to affect the necessary number for an average. How is this risk to be valued? In fact, the logical conclusion from this reasoning would be that members had no right to break their contracts, and therefore ought to get no value whatever. In practice the best answer to these objections would be that the liberality of the office in making no deduction on this account would by attracting more entrants give old members a greater profit than the loss from this source would take away.

I have now gone over all the objections I can remember that Mr. Sprague has brought, with one exception; and that one I have reserved for the conclusion because it has been the often iterated charge brought by him against the paper—viz., that it is unpractical. If it is called unpractical because it is based on theory, I at once admit the charge, but if by practical is meant the power of being worked in the great majority of cases, readily, safely, and advantageously under existing conditions, a valuation by a high rate of interest on a pure basis seems to me practical enough, and I think an inspection of these columns of the 5 and 6 per-cent interest values as compared with the empirical values will bear out the assertion.

On Mathematical Statistics and its application to Political Economy and Insurance. By Dr. Theodor Wittstein, *Actuary of the Hanover Life Insurance Company. Translated by* T. B. Sprague, M.A.

(Continued from page 369.)

§ 22.

We have now, in conclusion, to deduce from the foregoing investigations the proper formulas for deducing a mortality table from observations arranged as in § 17. The preceding notation will not be strictly adhered to.

1. For the most probable values $p, p', p'', \ldots.$ of the probabilities of living a year, and the probable errors, $s, s', s'', s''' \ldots.$ of these probabilities for successive years of age, x, $x+1$, $x+2$, $x+3, \ldots.$, we have the formulas,

$$\left.\begin{array}{ll} p=\dfrac{\mathrm{L}'}{\mathrm{L}} & s=\cdot 6745\sqrt{\dfrac{p(1-p)}{\mathrm{L}}} \\ p'=\dfrac{\mathrm{M}'}{\mathrm{M}} & s'=\cdot 6745\sqrt{\dfrac{p'(1-p')}{\mathrm{M}}} \\ p''=\dfrac{\mathrm{N}'}{\mathrm{N}} & s''=\cdot 6745\sqrt{\dfrac{p''(1-p'')}{\mathrm{N}}} \\ \&c. = \&c. & \&c. = \&c. \end{array}\right\} \quad . \quad . \quad . \quad (48)$$

whence, according to well known propositions, we get the values of the probabilities ${}_2p, {}_3p, \ldots.$ of being alive at the end of 2, 3, ... years, and their probable errors, $s_2, s_3 \ldots.$:—

$$\left.\begin{array}{ll} {}_2p=pp' & s_2^2=s'^2p^2+s^2p'^2 \\ {}_3p={}_2pp'' & s_3^2=s''^2{}_2p^2+s_2^2p''^2 \\ \&c.=\&c. & \&c.=\&c. \end{array}\right\} \quad . \quad . \quad . \quad (49)$$

2. For the most probable values of the numbers, $l', l'', l''', \ldots.$ alive at successive ages, l being given or assumed at pleasure, and their probable errors, $\sigma', \sigma'', \sigma''', \ldots.$ we have the formulas,

$$\left.\begin{array}{ll} l'=lp & \sigma'^2=\cdot 6745^2l'(1-p)\left(1+\dfrac{l}{\mathrm{L}}\right) \\ l''=l'p' & \sigma''^2=\cdot 6745^2l''(1-p')\left(1+\dfrac{l'}{\mathrm{M}}\right)+\sigma'^2p'^2 \\ l'''=l''p'' & \sigma'''^2=\cdot 6745^2l'''(1-p'')\left(1+\dfrac{l''}{\mathrm{N}}\right)+\sigma''^2p''^2 \\ \&c.=\&c. & \&c.=\&c. \end{array}\right\} \quad . \quad . \quad . \quad (50)$$

Or, if $s, s', s'', \ldots$ are already calculated,

$$\left.\begin{array}{ll} l' = lp & \sigma'^2 = s^2 l(\mathrm{L}+l) \\ l'' = l'p & \sigma''^2 = s'^2 l'(\mathrm{M}+l') + \sigma'^2 p'^2 \\ l''' = l''p & \sigma'''^2 = s''^2 l''(\mathrm{N}+l'') + \sigma''^2 p''^2 \\ \&c. = \&c. & \&c. = \&c. \end{array}\right\} \quad \ldots \quad (51)$$

If $l, l', l'', l''', \ldots$ are small in comparison with L, M, N, ... these formulas reduce to

$$\left.\begin{array}{ll} l' = lp & \sigma' = \cdot 6745 \sqrt{\dfrac{l'(l-l')}{l}} \\ l'' = l'p' & \sigma'' = \cdot 6745 \sqrt{\dfrac{l''(l-l'')}{l}} \\ l''' = l''p'' & \sigma''' = \cdot 6745 \sqrt{\dfrac{l'''(l-l''')}{l}} \\ \&c. = \&c. & \&c. = \&c. \end{array}\right\} \quad \ldots \quad (52)$$

Observation. In deducing the equations (49) a proposition has been made use of which when fully stated runs as follows:—

If we know a, the most probable value of a certain magnitude, with a probable error α; and b, the most probable value of another magnitude independent of the former, with a probable error β; and we desire to find γ, the probable error of the most probable value c of a third magnitude compounded of the other two; then we have

(1) If c is the sum or difference of a and b,

$$\gamma^2 = \alpha^2 + \beta^2;$$

(2) If c is the product or quotient of a and b,

$$\frac{\gamma^2}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}.$$

The proposition still holds good if we substitute throughout mean errors for probable errors.

§ 23.

If, when $p, p', p'', \ldots$ or $l, l', l'', \ldots$ have been calculated as above explained, the first differences do not proceed with sufficient regularity, this must be looked upon as a proof that the observations have not been numerous enough. But it is difficult to form a completely impartial judgement as to the conditions these differences should fulfil. In strictness the figures resulting from calculations based upon the preceding formulas should be used unaltered as the basis of all future applications; for every alteration must be more or less arbitrary and consequently disturb

the basis upon which these results rested. If, however, we wish to alter them for the purpose of producing more regular differences, with as little departure as possible from the observed facts, this can perhaps be best done as follows.

Let $a, b, c, \ldots$ be the terms of a series of which it is known beforehand that the first differences should change slowly. If now, in consequence of the series being deduced from given observations, this is not the case, but the differences change more or less abruptly, then let a new series be formed by taking the arithmetic means of every five consecutive terms of the series, that is to say, take

$$c' = \frac{a+b+c+d+e}{5}, \quad d' = \frac{b+c+d+e+f}{5}, \ldots .$$

In the case we are considering, this operation evidently amounts to supposing that some of the deaths which ought to have occurred at a certain age, have chanced to occur one or two years too early or too late; and the above process is calculated to remove this error. Let the series so obtained be similarly treated, so that we get a new series of which the terms are

$$e'' = \frac{c'+d'+e'+f'+g'}{5}, \ldots .$$

or, substituting the above values of c', &c.,

$$e'' = \frac{a+2b+3c+4d+5e+4f+3g+2h+i}{25}, \ldots .$$

In this new operation we have included the four preceding and the four following values. We have not however attributed the same weight to all those values, but have given the nine terms in the value of e'' the respective weights 1, 2, 3, 4, 5, 4, 3, 2, 1, which are greatest in the middle and decrease on each side, as they should from the nature of the case.

The probable errors of the terms of each new series are got in the same way as the terms themselves. If, for instance, $\alpha, \beta, \gamma, \ldots$ are the probable errors of the terms $a, b, c, \ldots$ then will the probable errors $\gamma', \delta', \ldots$ of the terms $c', d', \ldots$ be given by the equations

$$\gamma'^2 = \frac{\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2}{5}, \quad \delta'^2 = \frac{\beta^2+\gamma^2+\delta^2+\epsilon^2+\zeta^2}{5}, \ldots .$$

and the probable errors $\epsilon'', \ldots$ of the terms $e'', \ldots$ by the equations

$$\epsilon''^2 = \frac{\gamma'^2+\delta'^2+\epsilon'^2+\zeta'^2+\eta'^2}{5},$$

or $$\epsilon''^2 = \frac{\alpha^2+2\beta^2+3\gamma^2+4\delta^2+5\epsilon^2+4\zeta^2+3\eta^2+2\theta^2+\iota^2}{25}.$$

It is evident that instead of the number 5 which has been here used in forming the arithmetic mean, there is nothing to prevent our taking any other odd number, or even, with the necessary modification, an even number; but I have hitherto found 5 the most suitable for the purpose.

It will suffice in practice, when the supposition of § 21 is admissible, to apply the above method to the series of values of $p, p', \ldots$ and to deduce from the values so corrected, the values of $l, l', l'', \ldots$ together with their probable errors. This course has therefore been adopted in the following examples.

Observation. It is evident that when the operations here described are carried out, the first four and the last four terms of the series will be wanting. This is no disadvantage if the facts are obtained from the experience of an Insurance Company, for in that case these terms are generally deduced from too small a number of observations to be trustworthy. The terms in question can easily be supplied, if desired, by applying the same principle and putting for the first terms of the series,

$$a' = \frac{a+b+c}{3},\ b' = \frac{a+b+c+d}{4},$$

$$a'' = \frac{5a+4b+3c+2d+e}{15},\ b'' = \frac{4a+5b+4c+3d+2e+f}{19},$$

$$c'' = \frac{3a+4b+5c+4d+3e+2f+g}{22},\ d'' = \frac{2a+3b+4c+5d+4e+3f+2g+h}{24};$$

and in like manner for the last terms. This is the method adopted in the appended Table II (see § 25).

Chapter 3.

Application of the Theory to figures obtained by observation.

§ 24.

If a mortality table is to be constructed from statistical facts, the first requisite is that the facts should be compiled according to correct principles. These principles have been fully explained above; but it may be useful to illustrate them in detail by an example. I will choose for that purpose the experience of the Hanover Life Insurance Company, which has been compiled under my superintendence, and will give the whole process of deducing

the mortality table from the facts. For tho' the numbers furnished by that company are not large, yet, bearing in mind that the idea of large numbers is really only a relative one, there can be no objection to our applying general principles to them so as to give an example of what can be done even with an experience of moderate extent.

The experience of the Hanover Life Insurance Company exhibited in Table I, extended over 34 years, and was compiled as follows.

1. For each of the 34 years a list was drawn up with columns showing

The age;
„ number living at the beginning of the year;
„ „ who entered in the course „
„ „ who exited „ „
„ „ who died „ „

and under the column headed "age" all years from the youngest to the oldest were entered.

2. In each of these lists the columns "entered," "exited," "died," were filled up directly from the office books.

3. The numbers of the living at the beginning of the first office year 1831-2, when the company commenced business, were similarly taken direct from the office books.

4. The numbers of the living in the other office years were obtained by the following rule:—From the sum of the "living" and "entered" in any office year and for any age, subtract the sum of the "exited" and "died," and the difference will be the "living" at the beginning of the next office year, and for the next higher age.

The following extract from the lists will illustrate this process.

Males and Females.

Age.	1852–3.				1853–4.				1854–5.	
	Living.	Entered.	Exited.	Died.	Living.	Entered.	Exited.	Died.	Living.	&c.
46	80	4	2	2						
47	79	1	1	0	80	0	1	1		
48	93	5	1	2	79	2	3	0	78	
49	92	1	0	1	95	2	1	1	78	
50	94	2	1	2	92	3	0	1	95	
51	...	...	...	...	93	3	0	0	94	
52	...	...	...	...	...	...	.	...	96	

5. The 34 lists so obtained were combined in a single list, as directed in § 8, that is to say, by adding all the numbers occupying similar positions; and thus was obtained the appended Table I. In performing this process males and females were kept separate, and it was not until the end that the common list for males and females was deduced.

The age is throughout to be understood as indicating the age completed by the life assured in the course of the office year under consideration, and is therefore at once got by subtracting the office year of birth from the office year under consideration.

§ 25.

On account of the smallness of the numbers in Table I, the separate mortality of males and females could not well be determined from them, and therefore all the further calculations are based on the list of combined males and females. One reason for this limitation was that the only investigation of immediate interest to the company itself was a comparison of the actual with the expected mortality; and for that purpose, separation of the males and females was not necessary.

The results of the calculation are shown in Table II, and were obtained as follows.

The lists of males and females had first of all to be limited to the ages 25 to 85, because beyond those ages there were no deaths, and such cases cannot be subjected to calculation.

Then in order to be able to apply the calculations of § 9, it is to be observed that in the Hanover Life Insurance Company the exits have hitherto always taken place at the end of the office year, so that in the formulas of § 9, we have to put $F=0$. This being done, the probability of living a year at each age is got from equations (17) and (48). In Table II (col: 2) is given the probability of dying at each age, and in col: 3 its probable error according to (48).

The method of § 23 is then applied to the values so found, in order to get the graduated values of the probability of dying at each age (col: 4) and its probable error (col: 5). The commencement of this calculation, which may serve as an illustration to § 23 and to the *Observation* appended thereto, is as follows:—

Age.	Ungraduated probability of Dying in a Year.	Sum of every 5 in the previous column.	Sum of every 5, as before.	Divisor.	Graduated probability of Dying in a Year.
	0				
	0				
	0	·00971			
	0	·02206			
25	·00971	·03522	·16449	15	·01097
26	·01235	·04159	·22139	19	·01165
27	·01316	·05591	·26128	22	·01188
28	·00637	·06661	·28400	24	·01183
29	·01432	·06195	·30292	25	·01212
30	·02041	·05794	·30478	25	·01219
31	·00769	·06051	&c.	&c.	&c.
32	·00915	·05777			
33	·00894	&c.			
34	·01158				
&c.	&c.				

Lastly from these adjusted values were calculated by formula (52) the numbers living at successive ages out of 10,000 living at 25 (col: 6), and the probable errors of the same (col: 7).

For comparison with a known mortality table I have also given in Table II the numbers living at each age according to the Experience of the 17 English Offices, reduced to the same radix of 10,000 living at 25.

§ 26.

It will be understood that it is not my intention to recommend for practical use the mortality table here deduced, the important numbers of which are contained in cols: 4 and 6 of Table II. The experience on which it is based is not extensive enough, as appears at once from the magnitude of the probable error, which is never so small as $\frac{1}{10}$th of the probability of dying. Notwithstanding this, I will now apply this table to the following office year of the Hanover Life Insurance Company, 1865-6.

Table III exhibits in cols: 1 to 9 the experience of the company in the office year 1865-6, for males and females combined, not only in respect of persons but also in respect of sums assured. If the entries and exits are reduced to persons living at the beginning of the year, as explained in § 9, and then the total number so found to be living at the beginning of the year is multiplied into the corresponding probability of dying found from Table II col: 4, we get column 10; and if we pursue the same course with the sums assured, we get column 11. The sums of cols: 10 and 11 consequently represent the most probable number of deaths and the most probable amount of the claims to be expected in the office year 1865-6.

This calculation must be limited to the ages 25 to 85 because the mortality table, Table II, does not extend further.

If now with the same limitation we take in Table III the number living and the total sums assured at the beginning of the year, similarly modified for the entries and exits of the year, we can directly deduce the probable error both of the deaths and of the claims.

For* let l be the number living at the beginning of the year,
l_0 the most probable number of survivers at the end of the year;

then σ being the probable error in the number of the survivers, or, what is the same thing, of the dying, we have by § 4, equation (9),

$$\sigma = \cdot 6745 \sqrt{\frac{l_0(l-l_0)}{l}}.$$

Again, let S be the sum assured at the beginning of the year,
S_0 „ most probable amount of the sum assured at the end of the year;

then, Σ being the probable error of the sum assured at the end of the year, or what amounts to the same thing, of the claims, we have in the same way

$$\Sigma = \cdot 6745 \sqrt{\frac{S_0(S-S_0)}{l}}.$$

In the case before us the numerical values are as follows:

$l = 3095 \cdot 5$	persons	$S =$	£332,828
$l_0 = 3009 \cdot 23$	„	$S_0 =$	£323,307
$l - l_0 = 86 \cdot 27$	„	$S - S_0 =$	£9,521

whence $\sigma = 6 \cdot 18$ persons;

that is to say, it is an even bet that the number of deaths will lie between 80·09 and 92·45. In the same way,

$$\Sigma = £672 \cdot 6;$$

that is to say it is an even bet that the amount of the claims will lie between £8848·35 and £10,193·55.

For the purpose of comparison the same calculation is given in cols: 12 and 13 based on the 17 English Offices' Table, but without excluding the highest and lowest ages. The numbers in question are

* The process here given is only a rough one, because it throws together all years of age, but it is generally sufficient for the purpose.

$$
\begin{array}{rlrl}
l = & 3124{\cdot}5 \text{ persons} & S = & £335{,}055 \\
l_0 = & 3043{\cdot}03 \quad ,, & S_0 = & £326{,}024 \\
l - l_0 = & 81{\cdot}47 \quad ,, & S - S_0 = & £9{,}031
\end{array}
$$

whence

$$\sigma = 6{\cdot}01 \text{ persons}, \qquad \Sigma = £654{\cdot}8;$$

or the probable limits of the number of deaths are in this case 75·46 and 87·48 persons, and the probable limits of the amount of the claims £8376·2 and £9685·8.

We can also say that according to the former mortality table there corresponds to the probable limits of mortality a deviation of 7·1 percent above or below its expected most probable value, both in respect to the number of persons and the sums assured (these two numbers always giving nearly the same percentage). On the other hand, according to the 17 Offices' table, the probable limits of the mortality admit of a deviation of 7·3 percent above or below the expected most probable value.

The actual mortality during the office year, as shown in columns 8 and 9, has been considerably less than the expected. This is partly due to the circumstance that, in consequence of the recent great increase of the company's business, very nearly a third of the lives assured at the end of the office year 1865-6, consisted of persons who had entered in the last two years, and who had therefore recently passed a satisfactory medical examination. For the rest, however, this deviation is only one of the usual results brought about by chance.

§ 27.

For the purpose of giving an instance based on larger numbers, I have readjusted the experience of the Berlin Widows' Fund, which Brune published in the *Allgemeine Versicherungszeitung* for 1847, and from which he deduced his well known mortality tables.

The results of my calculation are given for Males in Table IV, and for females in Table V; and I have appended to both tables, for the purpose of comparison, Brune's adjusted numbers, reduced to a radix of 10,000 persons alive at 25. The probable error of the number of survivers has therefore reference only to those numbers (Col: 7).

As will be seen, the probable error of the probability of dying falls in both tables at the middle years of life to about 4 percent of that probability, and consequently the probabilities of death here calculated have a much greater weight than those in the former example.

TABLE I.—*Mortality Experience of the Hanover Life Insurance Company for the Years* 1831-2 *to* 1864-5.

Age.	Males.				Females.				Males and Females.			
	Living at the beginning of the Year.	Entered in the course of the Year.	Exited in the course of the Year.	Died.	Living at the beginning of the Year.	Entered in the course of the Year.	Exited in the course of the Year.	Died.	Living at the beginning of the Year.	Entered in the course of the Year.	Exited in the course of the year.	Died.
15	...	2	...	...	...	1	...	...	...	3	...	...
16	1	1	...	...	1	...	1	...	2	1	1	...
17	1	1	...	...	...	1	...	...	1	2	...	...
18	2	...	...	...	1	1	...	...	3	1	...	...
19	2	5	1	...	2	1	...	...	4	6	1	...
20	4	3	...	...	3	4	...	...	7	7	...	...
21	7	4	...	...	7	4	1	...	14	8	1	...
22	10	16	1	...	9	2	...	1	19	18	1	1
23	20	16	1	...	9	6	2	1	29	22	3	1
24	34	30	1	...	12	5	1	...	46	35	2	...
25	58	56	2	1	15	4	...	...	73	60	2	1
26	97	84	5	1	19	8	1	1	116	92	6	2
27	154	95	3	2	24	5	1	1	178	100	4	3
28	227	110	6	1	28	8	...	1	255	118	6	2
29	303	156	2	6	35	6	3	...	338	162	5	6
30	427	144	9	10	37	6	2	1	464	150	11	11
31	525	166	10	4	38	8	3	1	563	174	13	5
32	633	168	14	7	40	16	...	...	673	184	14	7
33	745	180	9	8	56	8	1	...	801	188	10	8
34	873	189	14	11	64	10	2	1	937	199	16	12
35	1000	188	12	15	71	11	1	2	1071	199	13	17
36	1118	200	11	9	75	9	1	1	1193	209	12	10
37	1252	183	15	18	81	13	3	...	1333	196	18	18
38	1351	175	22	16	89	13	3	1	1440	188	25	17
39	1436	166	20	23	96	13	4	...	1532	179	24	23
40	1486	171	12	21	103	15	2	1	1589	186	14	22
41	1577	152	10	25	107	9	5	2	1684	161	15	27
42	1630	147	14	27	108	19	1	2	1738	166	15	29
43	1688	143	13	27	120	12	1	1	1808	155	14	28
44	1725	114	14	22	127	7	1	1	1852	121	15	23
45	1740	126	15	27	126	11	1	...	1866	137	16	27
46	1749	111	12	26	130	16	...	1	1879	127	12	27
47	1761	89	14	21	141	19	2	1	1902	108	16	22
48	1767	103	20	32	151	15	1	1	1918	118	21	33
49	1760	97	16	32	159	18	1	1	1919	115	17	33

50	1740	85	15	23	167	11	1	4	1907	96	16	27
51	1740	71	12	22	169	13	2	5	1909	84	14	27
52	1703	59	9	31	167	21	2	8	1870	80	11	39
53	1663	69	5	27	170	6	...	1	1833	75	5	28
54	1613	54	9	43	169	12	...	4	1782	66	9	47
55	1524	38	10	29	170	11	...	4	1694	49	10	33
56	1458	47	9	38	167	7	...	5	1625	54	9	43
57	1396	30	6	32	167	11	1	2	1563	41	7	34
58	1326	39	8	35	165	13	...	6	1491	52	8	41
59	1252	31	3	47	170	15	4	7	1422	46	7	54
60	1169	25	4	49	169	9	...	2	1338	34	4	51
61	1074	...	6	31	169	...	...	6	1243	...	6	37
62	969	1	4	28	156	...	1	6	1125	1	5	34
63	855	...	5	35	146	...	1	6	1001	...	6	41
64	753	...	2	32	132	...	...	5	885	...	2	37
65	681	...	1	32	121	...	1	5	802	...	2	37
66	584	...	4	30	105	...	...	3	689	...	4	33
67	516	...	3	28	100	...	...	5	616	...	3	33
68	452	...	4	19	89	...	...	2	541	...	4	21
69	387	...	4	25	80	...	...	2	467	...	4	27
70	329	...	...	20	76	...	...	5	405	...	...	25
71	283	...	1	14	66	...	...	3	349	...	1	17
72	229	...	1	14	59	...	...	5	288	...	1	19
73	186	...	1	23	52	...	...	3	238	...	1	26
74	141	...	1	7	44	...	...	1	185	...	1	8
75	121	...	...	11	40	...	...	2	161	...	...	13
76	95	...	1	9	37	...	...	1	132	...	1	10
77	79	...	...	14	30	...	...	2	109	...	...	16
78	60	...	...	4	24	...	...	2	84	...	...	6
79	47	...	1	9	20	...	...	3	67	...	1	12
80	36	...	...	6	11	...	...	1	47	...	...	7
81	26	...	...	6	7	...	...	...	33	...	...	6
82	16	...	...	4	5	...	...	...	21	...	...	4
83	10	...	...	5	4	...	...	1	14	...	...	6
84	3	...	...	1	3	...	...	1	6	...	...	2
85	2	...	...	1	1	...	...	1	3	...	...	2
86	1	...	...	...	...	...	...	...	1	...	...	...
	51682	4140	427	1176	5511	433	58	142	57193	4573	485	1318

TABLE II.—*Mortality Table deduced from the Experience of the Hanover Life Insurance Company for the years* 1831-2 *to* 1864-5.

1	2	3	4	5	6	7	8	9
	UNGRADUATED.		GRADUATED.				ACCORDING TO 17 OFFICES' EXPERIENCE.	
AGE.	Probability of Dying in a Year.	Probable Error.	Probability of Dying in a Year.	Probable Error.	Numbers Living.	Probable Error.	Probability of Dying in a Year.	Numbers Living.
25	·00971	·00652	·01097	·00554	10000·0	...	·00777	10000
26	1235	585	1165	523	9890·3	7·0	789	9922
27	1316	509	1188	484	9775·0	10·0	801	9844
28	0637	303	1183	440	9659·0	12·2	814	9765
29	1432	391	1212	400	9544·6	14·1	828	9686
30	2041	411	1219	353	9428·8	15·7	842	9606
31	0769	231	1160	307	9313·8	17·1	858	9525
32	0915	232	1120	273	9205·8	18·2	875	9443
33	0894	212	1110	249	9102·8	19·3	892	9360
34	1158	224	1097	227	9001·6	20·2	910	9277
35	1452	236	1107	210	8902·8	21·1	929	9192
36	0771	164	1134	202	8804·4	21·9	948	9107
37	1258	199	1191	197	8704·4	22·7	969	9021
38	1108	180	1248	194	8600·8	23·4	991	8933
39	1418	198	1311	192	8493·6	24·1	·01013	8845
40	1308	187	1356	190	8382·2	24·8	1036	8755
41	1530	197	1410	191	8268·6	25·5	1061	8664
42	1593	198	1427	189	8152·0	26·2	1089	8573
43	1485	188	1413	185	8035·6	26·8	1125	8479
44	1203	168	1391	182	7922·2	27·4	1170	8384
45	1396	180	1396	181	7812·0	27·9	1221	8286
46	1390	179	1397	180	7703·0	28·4	1284	8184
47	1125	161	1404	180	7595·4	28·8	1352	8079
48	1669	194	1466	183	7488·8	29·2	1426	7970
49	1670	194	1513	186	7379·0	29·7	1506	7857
50	1381	178	1579	191	7267·4	30·1	1594	7738
51	1384	178	1667	197	7152·7	30·4	1690	7615
52	2042	218	1804	208	7033·5	30·8	1795	7486
53	1497	189	1908	216	6906·5	31·2	1909	7352
54	2590	251	2068	229	6774·7	31·5	2031	7211
55	1920	223	2237	242	6634·7	31·9	2166	7065
56	2603	264	2457	259	6486·3	32·2	2313	6912
57	2147	246	2644	275	6327·0	32·5	2468	6752
58	2703	281	2862	294	6159·7	32·8	2639	6586
59	3737	337	3088	314	5983·4	33·1	2825	6412
60	3764	349	3285	336	5798·7	33·3	3034	6231
61	2977	325	3442	360	5608·3	33·5	3261	6042
62	3021	344	3657	391	5415·3	33·6	3512	5845
63	4096	423	3931	428	5217·3	33·7	3784	5639
64	4181	454	4140	465	5012·2	33·7	4083	5426

TABLE II—(*continued*).

1	2	3	4	5	6	7	8	9
	UNGRADUATED.		GRADUATED.				ACCORDING TO 17 OFFICES' EXPERIENCE.	
AGE.	Probability of Dying in a Year.	Probable Error.	Probability of Dying in a Year.	Probable Error.	Numbers Living.	Probable Error.	Probability of Dying in a Year.	Numbers Living.
65	4614	500	4395	507	4804·8	33·7	4408	5204
66	4790	549	4679	557	4593·6	33·6	4761	4975
67	5357	612	4893	606	4378·6	33·5	5147	4738
68	3882	560	5064	661	4164·4	33·3	5563	4494
69	5782	728	5525	751	3953·5	33·0	6009	4244
70	6173	807	5864	835	3735·1	32·6	6493	3989
71	4871	777	6216	934	3516·0	32·2	7016	3730
72	6597	987	6673	·01053	3297·5	31·7	7580	3468
73	·10924	·01364	7404	1221	3077·4	31·1	8188	3206
74	·04324	1009	7702	1371	2849·6	30·4	8847	2943
75	·08074	1448	8602	1615	2630·1	29·7	9556	2683
76	·07576	1553	9698	1920	2403·9	28·8	·10318	2426
77	·14679	2286	·11111	2327	2170·8	27·8	·11147	2176
78	·07143	1895	·12375	2846	1929·6	26·6	·12044	1933
79	·17910	3160	·15218	3709	1690·8	25·3	·13006	1701
80	·14894	3503	·18136	5039	1433·5	23·6	·14041	1480
81	·18182	4529	·22851	7044	1173·5	21·7	·15144	1272
82	·19048	5780	·27516	8678	905·3	19·4	·16319	1079
83	·42857	8921	·33002	·10253	656·2	16·7	·17591	903
84	·33333	·12981	·37535	·11837	439·7	13·8	·18968	744
85	·66667	·18357	·43434	·13379	274·6	11·0	·20510	603
86	...	...	...	...	155·4	8·3	...	479

TABLE III.—*Application of the foregoing table to the Office year* 1865-6 *of the Hanover Life Insurance Company.*

1	2	3	4	5	6	7	8	9	10	11	12	13
	MALES AND FEMALES.								ACCORDING TO TABLE II, COL. 4. Most probable		ACCORDING TO TABLE II, COL. 8. Most probable	
AGE.	Living at the beginning of the Year.		Entered in the course of the Year.		Exited in the course of the Year.		Died in the course of the Year.					
	Persons.	£	Persons.	£	Persons.	£	Persons.	£	Number of Deaths.	Amount of Claims. £	Number of Deaths.	Amount of Claims. £
15	...	...	1	30	...	...	...	...	...	...	·0035	·104
16	1	75	...	...	...	...	...	...	...	...	·0070	·525
17	1	75	...	...	...	...	...	...	...	...	·0071	·533
18	...	...	3	315	...	...	...	...	...	...	·0107	1·119
19	...	...	1	75	...	...	...	...	...	...	·0036	·270
20	2	45	2	90	...	...	...	...	...	...	·0219	·657
21	...	...	2	90	...	...	...	...	...	...	·0074	·333
22	2	660	4	255	1	600	...	...	...	...	·0263	3·655
23	6	450	5	255	...	...	...	...	...	...	·0646	4·389
24	2	300	11	585	...	...	...	...	...	...	·0578	4·563
25	6	390	12	660	2	165	...	...	·1207	6·993	·0858	4·972
26	15	870	17	1365	2	105	...	...	·2621	17·475	·1778	11·850
27	22	1365	16	1080	2	120	...	...	·3445	21·918	·2320	14·760
28	17	975	22	2025	1	75	...	...	·3253	23·069	·2228	15·795
29	29	2145	33	2280	1	90	...	...	·5454	39·268	·3735	26·892
30	27	2550	29	2475	...	...	...	...	·5059	46·170	·3486	31·815
31	31	3345	28	1710	2	75	...	...	·5104	48·285	·3784	35·797
32	47	3885	37	2355	3	165	...	...	·7168	55·776	·5568	43·326
33	37	4200	29	2460	...	150	...	...	·5717	59·440	·4584	47·660
34	37	3765	22	1485	1	60	...	...	·5211	49·119	·4323	40·746
35	39	4575	31	2415	...	...	...	...	·6033	64·013	·5069	53·778
36	50	6465	20	1860	1	225	...	...	·6747	82·584	·5653	69·185
37	49	5595	20	1335	...	...	...	...	·7027	74·586	·5723	60·747
38	55	5640	29	2475	...	...	1	75	·8674	85·832	·6881	68·088
39	54	5505	37	3090	2	75	2	225	·9374	91·934	·7222	70·827
40	78	9150	31	3750	...	...	1	60	1·2679	149·499	·9724	114·660
41	57	4935	19	2220	...	...	...	...	·9377	85·235	·7049	64·077
42	66	6240	20	975	1	75	1	75	1·0774	95·466	·8230	72·921
43	56	6150	25	2430	...	...	1	15	·9679	104·067	·7741	83·224
44	69	7620	13	1455	1	15	1	150	1·0433	116·010	·8775	97·578
45	75	7020	28	1830	1	150	...	...	1·2355	109·725	1·0797	95·892
46	78	10035	22	2280	1	60	...	...	1·2364	155·696	1·1328	142·656
47	66	8580	18	1590	1	75	...	...	1·0460	131·098	1·0058	126·057
48	54	5670	11	1005	...	...	...	...	·8723	90·489	·8509	88·268
49	64	7470	13	870	1	30	...	...	1·0591	119·376	1·0570	119·139
50	78	7470	7	375	...	...	1	75	1·2869	120·912	1·2959	121·755
51	52	5490	5	570	...	...	1	15	·9085	96·270	·9211	97·597
52	83	8550	6	465	...	...	1	150	1·5514	158·436	1·5394	157·208
53	67	8925	3	240	...	...	2	600	1·3070	172·578	1·3084	172·759
54	94	9615	5	225	1	150	3	345	1·9853	199·614	1·9488	195·947

TABLE III—(*continued*).

1	2	3	4	5	6	7	8	9	10	11	12	13
	MALES AND FEMALES.								ACCORDING TO TABLE II, COL 4.		ACCORDING TO TABLE II, COL. 8.	
AGE.	Living at the beginning of the Year.		Entered in the course of the Year.		Exited in the course of the Year.		Died in the course of the Year.		Most probable		Most probable	
	Persons.	£	Persons.	£	Persons.	£	Persons.	£	Number of Deaths.	Amount of Claims. £	Number of Deaths.	Amount of Claims. £
55	98	11880	3	255	...	...	2	210	2·2258	268·608	2·1592	260·563
56	75	8550	1	165	2	165	2	240	1·8305	210·074	1·7210	197·505
57	65	7890	1	675	...	...	1	75	1·7318	217·534	1·6179	203·220
58	75	8490	1	15	...	...	2	600	2·1608	243·198	1·9932	224·334
59	73	8955	...	300	1	15	6	765	2·2388	280·931	2·0445	256·550
60	69	8655	...	...	1	75	1	150	2·2502	283·084	2·0756	261·109
61	74	7830	...	...	...	...	2	300	2·5471	269·508	2·4124	255·258
62	75	8355	...	...	...	...	4	405	2·7428	305·543	2·6325	293·261
63	86	10335	...	...	1	150	2	390	3·3610	403·320	3·2319	387·828
64	69	7560	...	...	...	...	1	15	2·8566	312·984	2·8152	308·448
65	44	5430	...	...	...	...	2	105	1·9338	238·648	1·9404	239·463
66	74	7755	...	...	...	...	6	750	3·4625	362·856	3·5224	369·138
67	36	4125	...	...	...	...	1	45	1·7615	201·837	1·8540	212·438
68	39	4740	...	...	...	...	2	180	1·9750	240·033	2·1584	263·544
69	49	6540	...	...	...	...	1	75	2·7073	361·335	2·9449	393·054
70	31	3645	...	...	...	...	1	195	1·8178	213·742	2·0119	236·560
71	31	3510	...	...	...	...	1	15	1·9270	218·181	2·1762	246·402
72	43	4920	...	...	...	...	1	45	2·8694	328·311	3·2594	372·936
73	30	3780	...	...	...	...	2	195	2·2212	279·872	2·4570	309·582
74	26	2040	...	...	...	...	1	90	2·0025	157·120	2·3010	180·540
75	15	1155	...	...	...	...	3	150	1·2903	99·353	1·4340	110·418
76	16	1560	...	...	...	...	1	300	1·5517	151·288	1·6512	160·992
77	12	1665	...	...	...	...	1	45	1·3333	184·998	1·3380	185·647
78	9	630	...	...	...	...	1	120	1·1138	77·962	1·0836	75·852
79	11	1725	...	...	...	...	...	...	1·6740	262·511	1·4311	224·423
80	7	750	...	...	...	...	...	...	1·2695	136·020	·9828	105·300
81	7	600	...	...	...	...	1	75	1·5996	137·106	1·0598	90·840
82	6	720	...	...	...	...	...	...	1·6510	198·115	·9792	117·504
83	3	465	...	...	...	...	3	465	·9901	153·459	·5277	81·794
84	2	105	...	...	...	...	...	...	·7507	39·412	·3794	19·918
85	1	30	...	...	...	...	...	...	·4343	13·031	·2051	6·153
86	..	...	...	...	...	...	...	...	...	...	...	...
87	1	75	...	...	...	...	...	...	...	...	·2422	18·165
	Total for all ages from 25 up to 85.											
	2803	308580	614	50760	29	2265	66	7785	86·2737	9520·907	...	
	Total for all ages from 15 up to 87.											
	2818	310260	643	52455	30	2865	66	7785	...	...	81·4659	9030·863

Table IV.—*Mortality Table for Males, deduced from the Experience of the Berlin Widows' Fund from* 1776 *to* 1845.

1	2	3	4	5	6	7	8	9
	Ungraduated.		Graduated.				According to Brune's Table.	
Age.	Probability of Dying in a Year.	Probable Error.	Probability of Dying in a Year.	Probable Error.	Numbers Living.	Probable Error.	Probability of Dying in a Year.	Numbers Living.
21	·02564	·01707						
22	0873	·00586						
23	1364	456						
24	1012	256						
25	·00268	090	·00831	·00430	10000·0	...	·00676	10000
26	788	115	728	211	9917·0	6·1	681	9932
27	624	·00081	691	140	9844·8	8·3	685	9865
28	498	60	675	·00090	9776·8	10·0	701	9797
29	968	72	708	70	9711·0	11·3	718	9728
30	637	52	742	62	9642·3	12·5	734	9659
31	642	48	771	56	9570·6	13·7	763	9588
32	916	53	819	22	9496·8	14·7	804	9515
33	854	48	862	50	9419·2	15·8	857	9438
34	933	48	899	49	9338·0	16·8	900	9357
35	915	46	945	48	9254·0	17·7	932	9273
36	938	46	993	48	9166·6	18·6	977	9189
37	·01091	49	·01038	48	9075·8	19·5	·01035	9097
38	1068	48	1080	48	8981·6	20·4	1095	9003
39	1123	49	1128	49	8884·4	21·2	1157	8904
40	1251	51	1184	50	8784·2	22·0	1209	8801
41	1131	48	1236	51	8680·0	22·8	1249	8695
42	1211	50	1299	52	8572·8	23·6	1291	8586
43	1429	55	1378	54	8461·4	24·3	1347	8475
44	1521	57	1453	56	8344·7	25·1	1405	8361
45	1478	57	1524	58	8223·6	25·8	1479	8244
46	1580	60	1601	61	8098·2	26·5	1555	8122
47	1761	64	1668	63	7968·8	27·1	1649	7996
48	1680	64	1729	66	7836·0	27·8	1747	7864
49	1811	68	1803	69	7700·5	28·4	1836	7726
50	1874	71	1891	72	7561·7	29·0	1914	7584
51	1774	71	1992	77	7418·7	29·5	2011	7439
52	2168	81	2126	82	7271·0	30·0	2113	7290
53	2221	85	2275	87	7116·3	30·6	2236	7136
54	2474	93	2432	94	6954·6	31·0	2367	6976
55	2614	99	2605	·00101	6785·5	31·5	2522	6811
56	2766	·00105	2779	108	6608·7	31·9	2704	6639
57	2957	113	2967	116	6425·1	32·3	2882	6460
58	2964	119	3169	126	6234·4	32·7	3091	6274
59	3530	135	3411	136	6036·9	33·0	3335	6080
60	3397	139	3674	149	5831·0	33·3	3620	5877

TABLE IV—(*continued*).

1	2	3	4	5	6	7	8	9
	UNGRADUATED.		GRADUATED.				ACCORDING TO BRUNE'S TABLE.	
AGE.	Probability of Dying in a Year.	Probable Error.	Probability of Dying in a Year.	Probable Error.	Numbers Living.	Probable Error.	Probability of Dying in a Year.	Numbers Living.
61	4074	159	3976	162	5616·7	33·5	3952	5664
62	4115	168	4297	178	5393·4	33·6	4297	5440
63	4702	189	4687	196	5161·7	33·7	4639	5207
64	5129	209	5086	216	4919·7	33·7	4977	4965
65	5154	222	5465	237	4669·5	33·7	5308	4718
66	5839	250	5868	261	4414·4	33·5	5655	4468
67	7010	291	6354	291	4155·4	33·2	6073	4215
68	6664	306	6844	324	3891·4	32·9	6577	3959
69	6221	318	7425	365	3625·0	32·4	7130	3699
70	7494	372	8154	414	3355·9	31·8	7774	3435
71	9707	454	8869	467	3082·3	31·1	8465	3168
72	·10004	504	9543	529	2808·9	30·3	9285	2900
73	·10894	576	·10135	596	2540·9	29·4	·10194	2630
74	·11024	640	·10555	670	2283·4	28·3	·11116	2362
75	·09460	670	·11021	765	2042·4	27·2	·12032	2100
76	·12558	844	·11631	884	1817·3	26·0	·12598	1847
77	·10976	894	·12312	·01031	1605·9	24·8	·12903	1614
78	·10793	992	·13543	1238	1408·2	23·5	·13081	1406
79	·16997	·01348	·15197	1498	1217·5	22·1	·13509	1222
80	·15014	1495	·16582	1814	1032·5	20·5	·14961	1057
81	·17187	1836	·17990	2194	861·3	18·9	·15667	905
82	·25695	2456	·19837	2723	706·4	17·3	·17562	763
83	·21266	2777	·21135	3385	566·2	15·6	·20070	629
84	·16236	3391	·21829	4127	446·6	13·9	·22908	503
85	·21164	4008	·22937	5124	349·1	12·4	·25429	388
86	·32787	5733	...	...	269·0	10·9	...	289
87	·26667	6888						
88	·16667	7256						
89	·25806	·10602						

TABLE V.—*Mortality Table for Females, deduced from the Experience of the Berlin Widows' Fund from* 1776 *to* 1845.

1	2	3	4	5	6	7	8	9
	Ungraduated.		Graduated.				According to Brune's Table.	
Age.	Probability of Dying in a Year.	Probable Error.	Probability of Dying in a Year.	Probable Error.	Numbers Living.	Probable Error.	Probability of Dying in a Year.	Numbers Living.
16	·00826	·00555						
17	2098	467						
18	1761	263						
19	1536	169						
20	1405	125	·01444	·00221	10690·0	...	·01406	10670
21	1351	100	1387	156	10535·5	...	1339	10520
22	1163	·00080	1306	108	10389·2	...	1281	10380
23	1229	73	1259	·00085	10253·5	...	1231	10247
24	1268	67	1230	72	10124·5	...	1190	10120
25	1189	60	1209	64	10000·0	...	1159	10000
26	1290	59	1193	59	9879·2	7·4	1149	9884
27	1072	51	1176	55	9761·4	10·3	1151	9771
28	1136	50	1164	52	9646·5	12·5	1165	9658
29	1189	50	1160	51	9534·0	14·2	1166	9546
30	1111	47	1158	49	9423·4	15·7	1168	9434
31	1208	49	1165	48	9314·3	17·0	1182	9324
32	1107	46	1174	48	9205·8	18·2	1184	9214
33	1240	48	1189	48	9097·6	19·3	1198	9105
34	1171	47	1203	48	8989·4	20·3	1200	8996
35	1238	48	1216	48	8881·4	21·3	1202	8888
36	1193	47	1219	48	8773·4	22·1	1203	8781
37	1290	50	1226	49	8666·4	22·9	1218	8675
38	1299	50	1219	49	8560·4	23·7	1220	8570
39	1139	47	1201	49	8456·0	24·4	1208	8465
40	1133	48	1186	49	8354·5	25·0	1196	8363
41	1242	51	1184	50	8255·4	25·6	1183	8263
42	1152	50	1186	51	8157·6	26·1	1183	8165
43	1077	50	1197	53	8060·8	26·7	1183	8069
44	1190	54	1231	55	7964·3	27·2	1197	7973
45	1339	58	1262	57	7866·5	27·6	1226	7878
46	1380	61	1293	59	7767·2	28·1	1270	7781
47	1260	60	1317	62	7666·7	28·5	1301	7682
48	1428	66	1362	65	7565·7	28·9	1349	7582
49	1161	61	1406	68	7462·6	29·4	1397	7480
50	1547	72	1479	71	7357·7	29·7	1463	7376
51	1339	69	1561	76	7248·8	30·1	1532	7268
52	1894	84	1681	81	7135·8	30·5	1619	7156
53	1757	84	1794	86	7015·8	30·9	1727	7040
54	1872	90	1935	92	6890·0	31·2	1872	6919
55	1950	94	2084	99	6756·8	31·6	2058	6789

TABLE V—(*continued*).

1	2	3	4	5	6	7	8	9
	UNGRADUATED.		GRADUATED.				ACCORDING TO BRUNE'S TABLE.	
AGE.	Probability of Dying in a Year.	Probable Error.	Probability of Dying in a Year.	Probable Error.	Numbers Living.	Probable Error.	Probability of Dying in a Year.	Numbers Living.
56	2255	·00104	2252	·00106	6615·9	31·9	2238	6650
57	2380	111	2421	114	6467·0	32·2	2429	6501
58	2751	123	2622	122	6310·4	32·5	2615	6343
59	2799	128	2813	131	6145·0	32·8	2777	6177
60	2819	134	3014	140	5972·1	33·1	2951	6005
61	3223	148	3236	151	5792·1	33·3	3138	5828
62	3639	162	3488	162	5604·8	33·5	3361	5645
63	3496	166	3771	176	5409·3	33·6	3644	5456
64	3997	184	4121	191	5205·4	33·7	3998	5257
65	4191	197	4504	209	4990·9	33·7	4412	5047
66	4957	222	4961	229	4766·1	33·7	4899	4824
67	5663	248	5459	252	4529·7	33·6	5424	4588
68	5906	266	5993	277	4282·4	33·4	5970	4339
69	6041	284	6557	305	4025·8	33·1	6544	4080
70	7406	329	7179	338	3761·8	32·7	7122	3813
71	7717	357	7797	373	3491·8	32·2	7700	3541
72	8570	398	8481	415	3219·6	31·5	8342	3269
73	9077	439	9175	462	2946·6	30·7	9063	2996
74	9552	481	9861	514	2676·3	29·9	9800	2724
75	·10510	542	·10563	575	2412·4	28·9	·10541	2457
76	·12119	626	·11219	642	2157·6	27·7	·11215	2198
77	·11719	678	·11846	722	1915·5	26·5	·11758	1952
78	·12075	753	·12622	821	1688·6	25·3	·12269	1722
79	·13439	875	·13528	943	1475·5	23·9	·12857	1511
80	·12478	935	·14444	·01083	1275·9	22·5	·13719	1317
81	·15801	·01145	·15614	1255	1091·6	21·0	·15100	1136
82	·18579	1371	·16878	1456	921·1	19·5	·16843	965
83	·18315	1579	·17897	1690	765·7	17·9	·18555	802
84	·17593	1747	·18933	1979	628·7	16·4	·19826	653
85	·20809	2082	·19834	2306	509·6	14·8	·20607	524
86	·21805	2415	·20511	2715	428·9	13·7	·21038	416
87	·19318	2839	·21516	3369	341·0	12·2	·21107	328
88	·25000	3542	·23147	4052	267·6	10·9	·21053	253
89	·18367	3731	·24399	4974	205·7	9·6	·21667	204
90	·23529	4907	·26029	6114	155·5	8·3	·23404	160
91	·36000	6475	·27408	7354	115·0	7·2	·25926	123
92	·35714	8638	·28780	8891	83·5	6·1	·28750	91
93	·22223	9347	·30067	·11005	59·5	5·2	·33333	65
94	·28571	·11517	...	...	41·6	4·3	...	43
95	·20000	·12066						
96	·50000	·16862						
97	·50000	·23847						

HOME AND FOREIGN INTELLIGENCE.

THE LONDON ASSURANCE CORPORATION.

Established 1720.

QUINQUENNIAL VALUATION, 31ST DECEMBER, 1870.

BALANCE SHEET, 31st December, 1870.

LIABILITIES.

				£	s.	d.
Shareholders' Capital ...				448,275	0	0
General Reserve Fund ...				270,592	2	3
Life Assurance Funds ...				1,378,822	14	5
Fire Fund ...				118,249	12	8
Marine Fund ...				233,061	4	8
Profit and Loss ...				107,499	17	0
				2,556,500	11	0
Claims under Life Policies admitted but not yet paid ...	32,128	2	0			
Outstanding Fire Losses ...	661	14	4			
Do. Marine do. ...	2,998	10	2			
Do. Annuities ...	170	2	4			
Do. Abatement of Premium ...	7	9	10			
Do. Dividends to Shareholders	197	10	0			
Do. Income Tax ...	210	10	4			
Clerks' Savings' Fund ...	2,164	11	7			
				38,538	10	7
				2,595,039	1	7

ASSETS.

				£	s.	d.
Mortgages on property within the United Kingdom...				1,460,920	13	4
Do. do. out of the United Kingdom ...				Nil.		
Loans on the Corporation's Life Policies ...				35,128	0	5
Investments, viz.:—						
In British Government Securities ...				447,241	10	10
„ Indian and Colonial do. ...				238,321	18	4
„ Foreign do., viz.: Turkish Bonds 4 per Cent. Guaranteed ...				24,800	0	0
„ Railway and other Debentures ...				253,700	0	0
„ Do. Shares ...				Nil.		
„ House Property ...				Nil.		
„ Reversions ...				16,861	17	7
„ Government Life Annuities (£1,981. 3s. per annum) ...				8,307	0	0
Loans upon personal security ...				Nil.		
Agents' balances ...				28,460	3	10
Outstanding Premiums ...				16,166	5	9
Do. Interest ...				3,858	4	0
Cash, viz.:—						
On deposit ...	40,000	0	0			
In hand and on current account ...	14,378	3	4			
				54,378	3	4
Bills receivable ...				6,688	11	8
Policy Stamps (balance of account) ...				206	12	6
				2,595,039	1	7

The following further particulars are taken from the Returns under the Fifth and Sixth Schedules of the Life Assurance Companies Act or from the Actuary's Report:—

The Life Assurance business of the Corporation is carried on under three distinct Series—the Old Series, the Series of 1831, and the Series of 1846. These Series are worked independently, as if they were distinct Companies, separate accounts being kept, and separate investments made for each.

Under the Old Series the assured do not participate in the Profits, and the principles on which the valuation is made are at the discretion of the Court of Directors.

Under the Series of 1831, the principles are determined by the contract made with the early Policy holders of that Series, which directs—

"That upon, or as up to, the thirty-first day of December, 1836, and afterwards yearly, and every year upon, or as up to, the thirty-first day of December in each successive year, an account and valuation shall be taken, shewing on the one hand, the computed amount or value of the claims and liabilities on the said Fund, including any advances which may have been found necessary in aid thereof to answer losses or otherwise; and on the other, the computed value of the said Fund, and of the Premiums payable during the expected remainder of the Lives assured; and that one-fifth, of whatever surplus may appear on taking such account, shall, on the first day of January next ensuing, be appropriated as follows; viz. two-thirds thereof, in abatement of the premiums, then due and payable by the assured then living and who shall have paid full five years' premiums, in proportion to the amount of their respective annual payments, and to be allowed on making such annual payments respectively; the remaining one-third shall be received and retained by the Corporation for their own use and benefit. Provided always, that the said abatement of premium to the assured, shall in no case exceed the amount of the annual premiums specified in the Policies; and should there be any surplus the same shall become the sole property of the Corporation."

The premiums on the policies of this Series are all due annually on the 1st January. A valuation of the Assets and Liabilities is made every year shortly before its close.

Under the Series of 1846, the principles are determined by the Court of Directors. The Two-Thirds of the Profits allotted to the Assured, are distributed in proportion to the amount of the Annual Premiums, exclusive of extra Premiums, paid during the quinquennium for which the Profits are divided. The share thus allotted to each Policy may, if desired, be converted either into an equivalent reversionary addition to the sum assured, or an equivalent reduction of the future Annual Premium.

The Table or Tables of Mortality used in the Valuation are

For the Old Series—The Seventeen Offices' Experience for the Assurances—the Government Annuitants' (1860) for the Annuities.

For the Series of 1831—According to the terms of the contract, the Government Male Annuitants' (1829).

For the Series of 1846—The Seventeen Offices' Experience.

The Rates of Interest assumed in the calculations are

For the Old and 1846 Series—Four per Cent.

For the Series of 1831—Three per Cent., in conformity with the contract.

Under the Series of 1831, Five full Annual Premiums must have been paid in order to entitle a policy to share in the profits.

Under the Series of 1846, all Policies share that are in force at the date of the Valuation.

The total amount of profit made by the Company was

Under the Old Series £29,449.; including £15,600., being two-fifths of the Profit of the last quinquennium, not then divided.

Under the Series of 1831, £88,452. 1s. 3d., whereof £17,690. 8s. 3d. was divided.

Under the Series of 1846, £131,074., whereof £120,000. was divided.

Of this surplus, the Directors have the power to reserve any portion, not exceeding one-tenth, to be carried over to the next quinquennial period. As experience has shewn that the mortality among recently selected lives is less than among those that have been longer assured, and as rather more than 30 per cent. of the Policies in force have been effected within the last five years, a somewhat higher rate of mortality may reasonably be expected to prevail in future. By retaining out of the surplus £11,074. to meet this contingency, £120,000 will be divisible, *i.e.*, £80,000. to the Assured, and £40,000. to the Corporation. The Bonus to the Assured will then be equivalent to 23·9 per cent. of the amount of the premiums received in the quinquennium, or 25·45 per cent. of the premiums received on the Policies in force on the 31st December, 1870, all of which are entitled to participate.

The amount of Profit divided among the Policy holders, and the number and amount of the Policies which participated were

Under the Series of 1831, £11,793. 12s. 2d. (two-thirds of the above Surplus), among 1,052 Policies for £1,079,785., in abatement of the Premiums amounting to £32,760. 0s. 5d. being at the rate of 36 per cent. thereon.

Under the Series of 1846, £80,000. among 2,995 Policies for £2,220,546.

SPECIMENS OF BONUSES ALLOTTED TO POLICIES FOR £100.

SERIES OF 1831.

Age at Entry.	Policies in force for 5 years and upwards.
	£. *s.* *d.*
20	0 14 5
30	0 18 3
40	1 3 5
50	1 12 7

Under this Series, the Bonus is apportioned every year in abatement of the Premium falling due on the 1st January following.

SERIES OF 1846.

REVERSIONARY BONUSES.

Age at Entry.	Policies in force for 5 years.	10 years.	15 years.	20 years.	25 years.
	£. s. d.	£. s. d.	£. s. d.	£. s. d.	£. s. d.
20	7 0 0	6 9 0	5 18 0	5 8 0	4 18 0
30	7 9 0	6 16 0	6 4 0	5 13 0	5 3 0
40	7 19 0	7 5 0	6 13 0	6 2 0	5 13 0
50	9 5 0	8 10 0	7 18 0	7 8 0	7 0 0

The following are examples of the total additions to which Policies of £1,000 each will be entitled, on which twenty-five annual premiums have been paid:—

Age at Entry.	Total additions.
20	£244 14 0
30	342 11 0
40	375 10 0
50	412 5 0

CASH BONUSES.

Age at Entry.	Policies in force for 5 years.	10 years.	15 years.	20 years.	25 years.
	£. s. d.	£. s. d.	£. s. d.	£. s. d.	£. s. d.
20	2 11 0	2 11 0	2 11 0	2 11 0	2 11 0
30	3 4 0	3 4 0	3 4 0	3 4 0	3 4 0
40	4 2 0	4 2 0	4 2 0	4 2 0	4 2 0
50	5 15 0	5 15 0	5 15 0	5 15 0	5 15 0

BONUSES IN REDUCTION OF THE ANNUAL PREMIUM.

Age at Entry.	Policies in force for 5 years.	10 years.	15 years.	20 years.	25 years.
	£. s. d.	£. s. d.	£. s. d.	£. s. d.	£. s. d.
20	0 2 4	0 2 5	0 2 7	0 2 10	0 3 1
30	0 3 3	0 3 7	0 3 11	0 4 4	0 4 11
40	0 5 0	0 5 7	0 6 4	0 7 5	0 8 10
50	0 8 10	0 10 3	0 12 3	0 15 0	0 18 8

Consolidated Revenue Accounts.

OLD SERIES, for THREE Years, commencing 1st January, 1868, and ending 31st December, 1870.

	£.	s.	d.
Amount of Funds on 1st January, 1868,—the beginning of the period	419,287	19	2
Premiums (after deduction of re-assurance Premiums)	121,171	2	11
Consideration for Annuities granted	12,258	8	7
Interest and Dividends (less Income Tax)	47,103	3	11
Undivided Profit of the Quinquennial period ending 31st December, 1867, re-transferred from Profit and Loss Account	15,600	0	0
	615,420	14	7

	£.	s.	d.
Claims under Policies (after deduction of sums re-assured) ...	149,050	0	11
Surrenders...	8,069	13	1
Annuities ...	41,741	19	8
Commission ...	3,895	12	3
Expenses of management—charged to Profit and Loss Account ...	—		
Premiums returned ...	610	17	3
Bad Debts...	903	14	0
Amount of Funds on 31st December, 1870,—the end of the period, as per third Schedule ...	411,148	17	5
	615,420	14	7

Series of 1831, for One Year, commencing 1st January, 1870, and ending 31st December, 1870.

	£.	s.	d.
Amount of Funds on 1st January, 1870,—the beginning of the period ...	432,509	19	4
Premiums (after deduction of re-assurance Premiums)	35,722	13	10
Interest and Dividends (less Income Tax) ...	20,344	12	2
Portion of Abatement appropriated to Policies not renewed in 1870 ...	148	18	5
	488,726	3	9

	£.	s.	d.
Claims under Policies (after deduction of sums re-assured) ...	26,997	0	0
Surrenders...	1,316	5	0
Commission ...	792	14	11
Expenses of management—charged to Profit and Loss Account ...	—		
Premiums returned ...	129	6	3
Abatement of Premiums ...	11,795	17	7
Amount of Funds on 31st December, 1870,—the end of the period, as per third Schedule ...	447,695	0	0
	488,726	3	9

Series of 1846, for Five Years, commencing 1st January, 1866, and ending 31st December, 1870.

	£.	s.	d.
Amount of Funds on 1st January, 1866,—the beginning of the period ...	345,127	4	9
Premiums (after deduction of re-assurance Premiums)	334,637	1	0
Interest and Dividends (less Income Tax) ...	93,198	1	5
	772,962	7	2

	£.	s.	d.
Claims under Policies (after deduction of sums re-assured) ...	200,867	5	8
Surrenders...	7,352	18	1
Commission ...	13,878	3	9
Expenses of management—charged to Profit and Loss Account ...	—		
Law Charges* ...	140	13	2
Bad Debts...	1,501	13	1
Premiums returned ...	1,051	5	1
Bonus to Assured taken in Cash ...	26,733	16	10
Ditto in Abatement of Premium ...	1,457	14	6
Amount of Funds, 31st December, 1870,—the end of the period, as per third Schedule ...	519,978	17	0
	772,962	7	2

* Since the Year 1868, the Law Charges have been included in the Expenses of Management, and charged to the Profit and Loss Account of the Corporation.

Summary and Valuation of the Policies of the Old Series, *as at 31st December*, 1870.

Description of Transactions.	Particulars of the Policies for Valuation. Number of Policies.	Sums Assured and Annuities.	Office Yearly Premiums.	Net Yearly Premiums.	Valuation. Value for Assurances by the Seventeen Offices' Experience Table; for Annuities, by Mr. A. G. Finlaison's Tables (Report 1860), Interest 4 per Cent. Sums Assured and Annuities.	Office Yearly Premiums.	Net Yearly Premiums.	Net Liability.
ASSURANCES.								
Without Participation in Profits.								
For Whole Term of Life	1,778	1,162,476	36,282	28,487	610,885	395,072	306,432	304,453
Endowments	34	8,600	423	337	5,667	3,671	2,899	2,768
Joint Lives	4	960	46	35	658	380	289	369
Last Survivor	22	27,470	496	330	10,719	7,572	4,982	5,737
Contingent	30	26,584	407	299	3,497	2,751	2,129	1,368
Against Issue	7	30,000	—	—	1,138	...	...	1,138
Increasing Premiums	4	5,700	138	—	...	...	...	69
For short terms	79	74,525	1,280	—	...	...	...	640
Extra Premiums Payable	—	—	1,460	—	...	...	...	730
Total Assurances	1,958	1,336,315	40,532	—	632,564	409,446	316,731	317,272
Deduct Re-Assurances	—	114,950	3,512	2,439	44,206	37,382	29,509	14,697
Net Amount of Assurances	—	1,221,365	37,020	—	588,358	372,064	287,222	302,575
Deduct for "Alternative Arrears," amounting to £22,557. of part of the Premiums due on Asylum Policies	...	...	...	...	...	...	...	19,218
								283,357
Add for Claims known, but not proved ...	...	...	...	...	...	...	...	2,099
								285,456
ANNUITIES.								
Immediate	262	13,854	...	...	91,734	...	...	91,734
Deferred	2	80	37	36	625	228	228	397
Reversionary	57	4,051	1,198	958	14,073	12,451	9,961	4,112
Total of the Results	...	...	38,255	...	694,790	384,743	297,411	381,699

Summary and Valuation of the Policies of the Series of 1831, *as at* 31*st December*, 1870.

Description of Transactions.	Particulars of the Policies for Valuation.				Valuation. Value by the Government Annuitants' Table—Interest 3 per Cent.			
	Number of Policies.	Sums Assured.	Office Yearly Premiums.	Net Yearly Premiums.	Sums Assured.	Office Yearly Premiums.	Net Yearly Premiums.	Net Liability.
ASSURANCES. *With Participation in Profits.*								
For Whole Term of Life	1,168	1,209,585	37,289	31,093	754,685	458,579	382,453	372,232
Re-Assurances from other Series	6	20,000	588	500	11,297	8,550	7,292	4,005
Extra Premiums payable	...	...	128	...	...	...	...	...
Total Assurances	1,174	1,229,585	38,005	...	765,982	467,129	389,745	376,237
Deduct Re-Assurances	...	120,150	3,134	2,833	61,833	49,804	45,292	16,541
Net amount of Assurances...	...	1,109,434	34,871	...	704,149	417,325	344,453	359,696
Deduction for the difference between the Assets as estimated on the 1st November, 1870, and the actual Assets on the 31st December, 1870 ...	...	...	...	...	...	...	...	453 1 3
Total of the results	...	...	...	...	...	...	...	359,242 18 9

Summary and Valuation of the Policies of the Series of 1846, *as at* 31*st December*, 1870.

Description of Transactions.	Particulars of Policies for Valuation.				Valuation. Value by the Seventeen Offices' Experience Table—Interest 4 per Cent.			
	Number of Policies.	Sums Assured.	Office Yearly Premiums.	Net Yearly Premiums.	Sums Assured and Bonuses.	Office Yearly Premiums.	Net Yearly Premiums.	Net Liability.
ASSURANCES.								
With Participation in Profits.								
For Whole Term of Life...	2,905	2,278,976	70,134	50,469	1,034,730	912,753	644,055	390,675
Do. Limited number of Premiums	26	25,629	1,161	825	9,830	9,353	6,620	3,210
Do. Increasing and Decreasing Premiums ...	3	4,695	162	129	2,634	1,826	1,461	1,173
Joint Lives	31	19,483	944	662	12,118	8,672	6,023	6,095
Last Survivor	29	23,089	454	285	7,359	7,045	4,310	3,049
Contingent	1	499	12	9	162	83	78	84
Re-Assurances from other Series	24	73,750	2,530	2,035	33,283	30,493	23,921	9,362
Extra Premiums payable	...	...	1,844	...	...	...	...	922
Total Assurances	3,019	2,426,121	77,241	...	1,100,116	970,225	686,468	414,570
Deduct Re-Assurances	...	222,353	6,641	4,911	95,473	91,980	67,455	28,018
Net amount of Assurances	...	2,203,768	70,600	...	1,004,643	878,245	619,013	386,552
Add value of Bonus applied in reduction of Premium—£161. per annum ...	...	...	...	...	...	...	...	1,600
Claims known, but not proved ...	...	...	...	...	...	...	...	750
Total of the results	...	...	...	...	...	...	...	**388,902**

Valuation Balance Sheet, as at 31*st December*, 1870.

DR.

To net Liability under Assurance and Annuity transactions, Old Series (as per Summary Statement)	381,699 0 0	
To Surplus...	29,449 17 5	
		411,148 17 5
To net Liability under Assurance transactions, Series of 1831 (as per Summary Statement) ...	359,242 18 9	
To Surplus...	88,452 1 3	
		447,695 0 0
To net Liability under Assurance transactions, Series of 1846 (as per Summary Statement) ...	388,902 0 0	
To Surplus...	131,076 17 0	
		519,978 17 0
		1,378,822 14 5

CR.

By Life Assurance and Annuity Funds, as per Balance Sheet, viz.:—		
Old Series	411,148 17 5	
Series of 1831	447,695 0 0	
Series of 1846	519,978 17 0	
		1,378,822 14 5
		1,378,822 14 5

The average rate of Interest on the Investments of the three Series, at the close of each year since the last investigation was made, is as follows:—

Year.	Old Series.	Series of 1831.	Series of 1846.
	£. s. d.	£. s. d.	£. s. d.
1866	......	......	4 12 6 per Cent.
1867	......	......	4 13 0 „
1868	4 0 0 per Cent.	......	4 14 1 „
1869	4 0 0 „	......	4 14 8 „
1870	4 6 1 „	4 10 5 per Cent.	4 13 9 „

Prior to 1870 there were no separate investments for the "Old Series," the Fund was charged against the general assets of the Corporation, interest at the rate of 4 per cent. being credited annually thereon.

The *minimum* value allowed for the surrender of assurances for the whole Term of Life, and for Endowments and Endowment Assurances, on which three or more annual Premiums have been paid, is 25 per cent. of the amount of the Premiums received thereon exclusive of extra Premiums. The *actual* value allowed is the result of a calculation made for each particular case.

Policies issued at other than European rates, are subject to the condition, that on the return home of the persons whose lives are assured, the Premiums are to be reduced to the European rates. They, therefore, form part of the ordinary risks, and are valued on the same principles, except that the amount of the extra Premium does not enter into the calculation.

In the case of Policies on unhealthy lives, the additional Premiums charged are regarded as a source of income, out of which the extra claims arising annually from the increased mortality of this class are defrayed. So that the "net liability" is the same, but the difference between the Office and net Premiums is greater, in these Policies than in similar Policies on healthy lives.

CORRESPONDENCE.

ON THE INTEGRAL OF GOMPERTZ'S FUNCTION.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—On reperusing my paper in the last Number of the *Journal*, I observe a slight misdescription of the capabilities of the table appended thereto, which I shall be glad to be allowed the opportunity of correcting.

The characteristic property of the gamma-function, expressed by the equation $\int \epsilon^{-v} v^m dv = -\epsilon^{-v} v^m + m \int \epsilon^{-v} v^{m-1} dv$, attaches also to the transformed expression $\int 10^{-10^z} \epsilon^{-nz} dz$; but in the latter case, the equation, instead of holding between successive integer values of n, or between successive values differing by *less than unity* (which latter supposition I had, by some means, erroneously entertained), applies to values of n proceeding by differences of $\log_e 10 (=2{\cdot}302 \ldots .)$. Consequently, in order to obtain the power of deducing the integral for *all* values of n, the tabulated matter must cover an interval equal to $2{\cdot}302 \ldots .$—that is to say, to be *theoretically* complete, the table must be extended in the ratio of $2{\cdot}302 \ldots .$ to 1.

The property in question may be readily demonstrated as follows:—

In $\int dz\, u \frac{dv}{dz} = uv - \int dz\, v \frac{du}{dz}$, the general formula for integration by parts, put

$$(1) \ldots . u = \left(\frac{1}{10}\right)^{10^z};$$

whence $$\frac{du}{dz} = -(\log_e 10)^2 . 10^{-10^z} . 10^z = -(\log_e 10)^2 10^{-10^z} \epsilon^{\log_e 10 . z}.$$

$$(2) \ldots . v = -\frac{\epsilon^{-nz}}{n};$$

whence $$\epsilon^{-nz}=\frac{dv}{dz};$$

giving,

$$\int 10^{-10^x}\epsilon^{-nz}dz=-10^{-10^x}\frac{\epsilon^{-nz}}{n}-\frac{(\log_e 10)^2}{n}\int 10^{-10^x}\epsilon^{-(n-\log_e 10)z}.dz.$$

I am, Sir,
Your very obedient servant,

London, 12 *June* 1873. W. M. MAKEHAM.

ERRATUM.—In the paper "On the Integral of Gompertz's Function," above referred to, there is a misprint. For the second formula on p. 309,

$$\log\frac{1}{g^{q^x}\epsilon^{-(a+\delta)x}}\int_x^\infty g^{q^x}\epsilon^{-(a+\delta)}.dx$$

should be read,

$$\log\frac{1}{g^{q^x}\epsilon^{-(a+\delta)x}}\int_x^\infty g^{q^x}\epsilon^{-(a+\delta)x}.dx.$$

ON THE RELATION BETWEEN THE NET PREMIUM AND THE RATE OF INTEREST.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In the current volume of the *Journal*, p. 227, I gave a demonstration in reference to the relation between the value of a policy and the rate of interest according to which it is calculated, and in the present letter I propose to examine in a similar way the relation between the net premium and the rate of interest.

We have $$P_x=\frac{1}{1+a_x}-(1-v),$$

$$\therefore\quad \frac{dP}{dv}=\frac{-\frac{da}{dv}}{(1+a)^2}+1 \text{ (omitting the subscript } x),$$

$$=\frac{(1+a)^2-\frac{da}{dv}}{(1+a)^2}.$$

Thus, since P_x increases or decreases, when v increases, according as $\frac{dP}{dv}$ is positive or negative, we have only to examine whether

$$(1+a)^2 > \text{ or } < \frac{da}{dv}.$$

Now, $(1+a)^2=(1+a_x)+p_xv(1+a_x)+{}_2p_xv^2(1+a_x)+\ldots.$

and as shown in my former letter,

$$v\frac{da}{dv} = a_x + p_x v a_{x+1} + {}_2p_x v^2 a_{x+2} + \dots$$

or

$$\frac{da}{dv} = \frac{a_x}{v} + \frac{p_x v a_{x+1}}{v} + \frac{{}_2p_x v^2 a_{x+2}}{v} + \dots$$

If, now, a_{x+1}, a_{x+2}, ... are none of them greater than a_x,

then

$$\frac{da}{dv} < \frac{a_x}{v} + \frac{p_x v a_x}{v} + \frac{{}_2p_x v^2 a_x}{v} + \dots$$

Hence, under the same condition, we shall certainly have

$$(1+a)^2 > \frac{da}{dv},$$

if

$$(1+a_x) + p_x v(1+a_x) + {}_2p_x v^2(1+a_x) + \dots$$

$$> \frac{a_x}{v} + \frac{p_x v a_x}{v} + \frac{{}_2p_x v^2 a_x}{v} + \dots$$

that is, if

$$1+a_x > \frac{a_x}{v},$$

or

$$i \cdot \frac{1}{i} + a_x > a_x + i a_x,$$

or

$$\frac{1}{i} > a_x;$$

that is, if the value of a perpetuity of 1 is greater than the value of a life annuity of 1, the rate of interest being the same in both cases.

In other words, since the value of the perpetuity is necessarily the greater, $\frac{dP}{dv}$ is positive; therefore P_x, the net premium, increases as the rate of interest decreases, provided that a_x is not less than a_{x+1}, a_{x+2}, ...

I am, Sir,
Your obedient servant,
W. SUTTON.

18 *Lincoln's Inn Fields,*
1 *March* 1873.

ON THE FORMULA FOR THE MARKET VALUE OF A COMPLETE ANNUITY.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—The usefulness of the expression for the value of a life annuity in terms of d and p, first proposed by the late Griffith Davies, is obvious, whether from a theoretical or practical point of view. From the theoretical, in that it shows the elements of which the value consists; and from the practical, in that it is of universal application, equally valid whether p and d be based on the same rate of interest or not, or when p is a purely arbitrary quantity.

In his work on Annuities, David Jones has introduced a modification of this expression, as a formula for determining the value of an annuity payable to the moment of death. I am not aware that the correctness of his formula has ever been called in question, and I recollect that it has, in at least one instance, been cited as correct in the pages of the *Journal* of the Institute. Notwithstanding this authority, however, and the long immunity of the formula from criticism, I think I shall be able to demonstrate that it is erroneous, and to show the correct expression which should be substituted for it.

The formula in question, which is given at p. 190 of Jones's work, and repeated at p. 217, is $\frac{1-p}{i+p}$. What may be called the rational basis on which this formula is constructed is this:—If the purchaser deducts from every 1 of outlay the annual premium for assuring that 1, and receives at the end of every completed year of the annuitant's life thereafter $i+p$, and the due proportion of $i+p$ for such part of the last year of life as shall have been lived through at the end of that year, these payments will have exactly yielded him interest on the 1 laid out and secured the repayment of the same. It is clear that up to the end of the last completed year of life the required conditions are fulfilled; interest and sinking fund are duly provided for—and it is *assumed* that the proportionate payment to be made in respect of the last uncompleted year will exactly meet the requirements of the case, neither exceeding nor falling short of them. But it is precisely this assumption that renders the formula erroneous. For what *are* the requirements? It must be borne in mind that, p being paid each year in advance, nothing more is wanted to complete the sinking fund after the payment at end of the last completed year; the assurance of 1 at end of the following year is secured, and all that the purchaser is then entitled to receive is i, the interest on his outlay for the year. But by the hypothesis on which the formula is based, what the purchaser will actually receive is the proportionate part of the payment he has annually received during the annuitant's life, viz., $\frac{i+p}{2}$. Therefore, when p is greater than i, he will receive more than he is entitled to; when p is less than i, he will receive less than he is entitled to; and only when p and i happen to be equal will he receive exactly his due.

Proceeding to construct a true formula, let I be the annual payment to be made in respect of 1 outlay. Then, since $\frac{\mathrm{I}}{2}$ will be received at the end of the year in which death occurs, in addition to ***such assured sum as together with it will make up the outlay***, it follows that this assured sum must be, not 1, but $1-\frac{\mathrm{I}}{2}$. From every 1 of outlay, then, the purchaser will deduct $d+p\left(1-\frac{\mathrm{I}}{2}\right)$; that is, a year's interest in advance and the premium required for assuring what is not otherwise secured to him of his outlay. In consideration of the price paid, $1-d-p\left(1-\frac{\mathrm{I}}{2}\right)$, he will receive an annual payment of I during

life, so that the value of an annuity of 1 during life with proportionate payment after death is $\dfrac{1-d-p\left(1-\frac{\mathrm{I}}{2}\right)}{\mathrm{I}}$. But from the equation $\mathrm{I}=d+p\left(1-\frac{\mathrm{I}}{2}\right)$ we can assign a value to I in terms of d and p, and we find that value to be $\dfrac{d+p}{1+\frac{p}{2}}$. Substituting this value in the foregoing formula, we have

$$\frac{1-d-p\left(1-\dfrac{\frac{d+p}{2}}{1+\frac{p}{2}}\right)}{\dfrac{d+p}{1+\frac{p}{2}}};$$

and this expression is finally reducible to the simple form $\dfrac{1-\left(d+\frac{p}{2}\right)}{d+p}$ which accordingly is the correct formula we have been seeking.

It is true that this formula might have been deduced with much more apparent simplicity from the old expression of Francis Baily $a_x+\dfrac{\mathrm{A}_x}{2}$; for not only a_x but A_x also is capable of being expressed in terms of d and p, so that this expression becomes $\dfrac{1-(d+p)}{d+p}+\dfrac{\frac{p}{2}}{d+p}$, or $\dfrac{1-\left(d+\frac{p}{2}\right)}{d+p}$; but I preferred constructing it on what I call a rational as distinguished from a purely formal basis, because it is not at once apparent whether the d and p of the second member of Baily's transformed expression are functions identical with those of the first member. From my construction it is evident that they are, and we thus arrive at the certitude which the expression requires to make it of universal application.

It will be found that the difference between the results obtained from Jones's formula and that which I propose to substitute for it, is too small to be of practical importance. I hope, however, it will not be thought useless to correct a theoretical error in a work which is so extensively used as a text book by the students of our profession.

I am, Sir,

Your obedient servant,

ANDREW BADEN.

1 *Old Broad Street, London.*
23 *June* 1873.

INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.—SESSION 1872-73.

First Ordinary Meeting, Monday, 25 *November* 1872.

The President in the Chair.

Read and confirmed the minutes of the anniversary meeting, held on 1st June 1872.

The following gentlemen were elected members, viz.:—

Fellows.

John Ralph Grimes.	John Montgomery Templeton.

Associates.

John Whitcher, Jun.	Thomas G. C. Browne.
Henry Hoskins.	Alfred B. Adlard.
Edward H. Carter.	Joseph M. Todd.
Alexander D. C. Clarke.	Allen Stoneham.
Edward E. Price.	Syman B. Cowin.
James M. Milligan.	Alexander S. Hume.
Thomas W. Stubbins.	

Mr. T. B. Sprague, M.A., read a paper "On Reversionary Life Interests as Securities for Loans."

Thanks having been voted to Mr. Sprague, the meeting adjourned to Monday, 30 December 1872.

Second Ordinary Meeting, Monday, 30 *December* 1872.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz.:—

William John Price.	Louis Michael Simon.
Wm. Evans.	Richd. M. Young.
Ernest Woods.	Wm. B. Macwhinnie.
J. J. W. Deuchar.	H. Woodrow, M.A.

Mr. Peter Gray read a paper "On the Arithmometer of M. Thomas (de Colmar), and its application to the Construction of Life Contingency Tables."

Thanks having been voted to Mr. Gray, the meeting adjourned to Monday, 27 January, 1873.

Third Ordinary Meeting, Monday, 27 *January* 1873.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz.:—

Capt. Augustus M. Festing.	William Saunders.
James Jackson.	Ernest W. Scott.

The following was announced to be the result of the Examinations for 1872:—

MATRICULATION EXAMINATION.

Nineteen gentlemen presented themselves for this Examination, and ten passed in the following order of merit:—

1. T. H. Cooke.
2. E. E. Price.
3. A. S. Hume.
4. A. B. Adlard.

Æq. { William Evans. / R. M. Moore.

7. W. G. Glennie.
8. W. Salter.

Æq. { J. J. W. Deuchar. / L. M. Simon, Jun.

SECOND YEAR'S EXAMINATION.

Fourteen gentlemen presented themselves for this Examination, and seven passed in the following order of merit:—

Æq. { F. A. Straker.
F. B. Wyatt.
3. Clarence Smith.
4. T. G. Ackland.
5. L. H. Greaves.
Æq. { Jas. H. Duncan.
F. Laing.

THIRD YEAR'S EXAMINATION.

One gentleman, Mr. John Duncan, presented himself for this Examination, and passed.

Mr. T. B. Sprague, M.A., opened a discussion on the 1st, 2nd, 3rd, and 4th schedules of the "Life Assurance Companies Act 1870," in which the following gentlemen took part, viz.:—Mr. John Coles, Mr. C. Walford, Mr. Grimes, Mr. Bailey, Mr. Peter Gray, Mr. Newbatt, Mr. H. A. Smith, Mr. Baden, and Mr. Messent.

Thanks having been voted to Mr. Sprague, the meeting adjourned to Monday, 24 February 1873.

Fourth Ordinary Meeting, Monday, 24 *February* 1873.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentleman was elected an Associate, viz.:—

William Henry Bedford.

Mr. Samuel Brown read a paper "On the application of the Binomial Law to Statistical Enquiries, illustrated by the Law of the growth of Man at different ages."

Thanks having been voted to Mr. Brown, the meeting adjourned to Monday, 31 March 1873.

Fifth Ordinary Meeting, Monday, 31 *March* 1873.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Associates, viz.:—

Rice Wasbrough. | John Davy.
Nicholson L. McEwan.

Mr. J. R. Macfadyen read a paper "On a general Formula for the value of present or future Benefits, whether free or burdened with charges; and on the application of the Formula to determining the Surrender Values of Life Policies."

Thanks having been voted to Mr. Macfadyen, the meeting adjourned to Monday, 28 April 1873.

Sixth Ordinary Meeting, Monday, 28 *April* 1873.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.

The following gentlemen were elected Fellows, viz.:—

Hippolyte Charlon. | J. S. Louis Onfroy.
Edmond Maas.

Mr. C. J. Bunyon, M.A., read a paper "On the Origin and Nature of some of these Limited and Contingent Interests in Property which are commonly submitted to Actuaries for Valuation."

Thanks having been voted to Mr. Bunyon, the meeting adjourned to Monday, 24 November 1873.

The Twenty-Fifth Annual General Meeting, Saturday, 7 June 1873.

ROBERT TUCKER, Esq., the President, in the Chair.

Mr. R. P. HARDY (Hon. Secretary) having read the circular calling the meeting, and the minutes of the last ordinary meeting, read the Report of the Council and the Statement of Accounts, which were as follows:—

"The Council have again the satisfaction of reporting an increase in the number of members of the Institute.

"Two Fellows and 29 Associates were elected during the year; and the total number on the books on the 31st March last, after allowing for deaths and withdrawals, was 313, of whom 107 were Fellows and 206 Associates. The total number at the close of the preceding year was 296. The annual subscriptions have reached the sum of £622 13*s.* 0*d.*, an increase upon the amount reported for the previous year of £26 5*s.* 0*d.*

"The income and expenditure of the Institute are stated in the accompanying account, certified by two of the Auditors, Mr. Hopkinson, the remaining Auditor, having been unable to attend on this occasion. The total funds amounted at the beginning of the year to £2,153 9*s.* 2*d.*, and at the end of the year to £1,880 1*s.* 2*d.* This decrease is to be attributed to the exhaustion of the Mortality Experience Fund in the publication of the monetary tables deduced from the Institute Experience. This work the Council have pleasure in reporting to have met with the general approval of the profession. The publication of the volume appeared to afford a fitting occasion for presenting Mr. Peter Gray with an arithmometer, in acknowledgment of the valuable assistance rendered by him to the Council in the production of the tables. The general expenditure has amounted to £631 15*s.* 6*d.*

"The following papers have been read during the past year:—

- 25 *Nov.* 1872.—'On Reversionary Life Interests as Securities for Loans.' By Mr. T. B. Sprague, M.A.
- 30 *Dec.* 1872.—'On the Arithmometer of M. Thomas (de Colmar), and its application to the Construction of Life Contingency Tables.' By Mr. Peter Gray.
- 27 *Jan.* 1873.—Mr. T. B. Sprague, M.A., opened a discussion on the 1st, 2nd, 3rd, and 4th Schedules of the 'Life Assurance Companies Act 1870.'
- 24 *Feb.* 1873.—'On the application of the Binomial Law to Statistical Enquiries, illustrated by the Law of the growth of Man at different ages.' By Mr. Samuel Brown.
- 31 *Mar.* 1873.—'On a general Formula for the value of present or future Benefits, whether free or burdened with charges; and on the application of the Formula to determining the Surrender Values of Life Policies.' By Mr. J. R. Macfadyen.
- 28 *April* 1873.—'On the Origin and Nature of some of these Limited and Contingent Interests in Property which are commonly submitted to Actuaries for Valuation.' By Mr. C. J. Bunyon, M.A.

"Having reason to be satisfied with the result of Mr. Sutton's re-appointment as Tutor, mentioned in their last report, the Council have again authorized the formation of a class for Second Year's Students under that gentleman's instruction. At the same time, it has been considered desirable that the Syllabus of all the Examinations should be revised. Having invited assistance from those gentlemen who had acted as examiners during the past ten years, the Council, after mature consideration, decided to adopt the new syllabus that has been circulated among the members.

"Several gentlemen holding a recognized position in the profession, in other countries, having expressed a desire to be allowed to join the Institute as contributing members, and a doubt having been expressed whether they

were eligible under the present provisions of the 'Constitution and Laws,' the Council, believing that such accessions to its ranks would further the objects of the Institute, recommend that such alterations be made in the 'Constitution and Laws' as will remove any doubt that persons resident out of the United Kingdom, being otherwise qualified, may be elected ordinary members."

The PRESIDENT said—"Gentlemen, it is my duty to move 'That the report of the Council, the abstract of income and expenditure, and the balance sheet be adopted, entered on the minutes, and printed in the *Journal*.' The accounts show that the financial position of the Institute continues to improve. There is an increase in the number of members and in the revenue. There is, however, a decrease in the amount of the funds, but that, as explained in the report, arises from the expense incident to the publication of the monetary tables deduced from the experience of the life assurance offices collected by the Council. The report, as usual, gives a *resumé* of our transactions during the past year. There is nothing very special to call for observation on my part. These reports are duly recorded, and whenever the history of the Institute comes to be written, ample materials will be found for the purpose in its archives. It is now about a quarter of a century since the Institute was projected. It may not be generally known that it had its origin in an attempt to form an association of actuaries and other managers of life offices similar to one which had been some little time in operation in Edinburgh. It was Mr. Thomson, of the Standard Office, who made this suggestion, and it fell to my lot to introduce Mr. Thomson to some of the leading actuaries of that day, with a view to the suggestion being considered. At a preliminary meeting held at the Standard Office, the late Mr. Peter Hardy took a more extended view of the scheme, and advocated the formation of a college or society of actuaries, on the model of other learned professions; and at a general meeting of the representatives of all the London offices, held subsequently, in the board-room of the Guardian, Mr. Hardy, Mr. Jellicoe and others, succeeded in carrying a resolution to that effect by what I may call an overwhelming majority. The effect of that resolution was the immediate formation of this Institute. It would not become me to speak of the leaders of the movement who still remain members of the Institute. Of that number, I regret to say that there are very few indeed left. But among those who have passed away there was one member whose name may not perhaps be so well remembered in connection with the Institute by the present generation as that of some others. I allude to the late Mr. Griffith Davies. (Hear, hear.) He cheerfully assisted in all our labours until the Institute was fairly afloat, and when it was understood that, with his characteristic unobtrusiveness, he declined to hold any office, the Council marked their sense of his services by presenting him with an address, written on vellum and beautifully illuminated. This address is preserved and highly cherished by his daughter, the only surviving lineal descendant of that distinguished actuary. It is no secret that in that day several actuaries did not join the Institute, and this defection continues to some extent at the present time. It is a source of deep regret to myself—and I know that regret is shared by others—that the profession is not, as it should be, wholly united. (Hear, hear.) I cannot help contrasting this want of sympathy with the cordial recognition our labours received from our professional brethren on the other side of the Tweed. It may be due to this want of unanimity that we have not obtained any legal status. Had the Institute been recognised by law, I cannot but think that much practical good would have resulted. (Hear, hear.) On a recent occasion, when the Life Assurance Bill was before the House of Commons, there was no authorised body of the profession to confer and advise with those who had charge of the bill. The consequence is, we have an Act of Parliament that is simply a nuisance to the sound offices, without being an efficient check upon the unsound ones. (Oh! and hear, hear.) Well, anybody who looks at the returns made under the fifth schedule, and

can deny what I say, is welcome to do it. (Laughter, and hear.) However, this is not the time to discuss the merits or demerits of the Life Assurance Companies Act. I merely refer to the measure as one instance tending to show how the usefulness of the Institute is restricted. At the same time I admit that if all that was aimed at has not been accomplished, much has been done in furtherance of its objects. The great services rendered by the Institute for educational purposes, and the high character it has attained by the publication of its *Journal* far surpass the most sanguine expectations of any of its founders. (Hear, hear.) The *Journal of the Institute*, verging on the completion of its seventeenth volume, has had among its contributors many great mathematicians, and it has been most fortunate in its editors. (Hear, hear.) The publication of the mortality experience of insurance offices and the volume of tables deduced from that experience recently published, partly at the expense of the Institute and partly by subscriptions from insurance companies, and the still more recent publication, at his own expense, by an active and enterprising member of this Institute, of a volume of useful tables—(hear, hear,)—will, in conjunction with the *Journal*, tend to popularize the Institute wherever life assurance is known. (Hear, hear.) I must apologize for occupying your time with these observations. If any member will do me the favour to second this resolution, I will, before it is formally put to the meeting, with the aid of my friend on my right (Mr. Hardy), reply to any questions arising out of the report and accounts which may be asked." (Cheers.)

Mr. T. B. SPRAGUE, M.A.—"I have great pleasure, Sir, in seconding the motion that the report be received and adopted and entered on the minutes. In doing so, I am not able to go back as you have done to the commencement of the Institute, as I only joined it at a period long subsequent to its formation, when perhaps it was at its lowest point. But ever since that time it has gone on prosperously; the members and the funds have continuously increased; and at the present time the only doubt in my mind is whether we are not getting rather too rich, and subject to all the evils that excessive wealth produces both in individuals and societies. (Hear, and laughter.) The report tells us that during the past year the funds of the Institute have suffered diminution, but I cannot help thinking that the diminution is more apparent than real. A sum of money has been paid to computers and to the printer for printing the volume of monetary tables, deduced from the Institute mortality tables; but that money is now represented by the stock of the book in question, for which stock we take no credit in our balance-sheet. If we had made out the account in the same way as a tradesman would have done, I think we should have found that our funds have rather increased than diminished. But even if the diminution had been real I think it would not have mattered to us. As I have mentioned on a previous occasion, it seems to me that we are accumulating money too fast, and that we really ought to consider whether we cannot do something else for the benefit of our members and the profession at large. (Hear, hear.) The Institute is already doing a great deal. The publication of the *Journal*, to which I myself am very pleased to contribute my humble services as editor—(cheers)—has been referred to, and it entails yearly considerable outlay upon the Institute. It is universally admitted that the study of the science on which our profession is based has been, and is still, very greatly advanced by that publication. (Hear, hear.) During the past few years the Institute has taken up another point of even greater practical importance, and that is the collection of the experience of the different insurance offices and the deduction of practical tables from it. That is a work of immense practical importance, the value of which is proved by the ready adoption of the table by a considerable number of companies;—I may say by the great majority of those which have made their valuation since its publication,—and it looks as if it would be universally accepted and adopted as the standard mortality table for our purposes, as it has not only been carefully compiled, but is free from the objections, such as they were, that

were brought against the old Experience Tables. (Hear, hear.) I will not trespass further upon your time, but simply second the adoption of the report."

The motion was carried unanimously.

Mr. W. Sutton, Mr. R. Wasbrough, and Mr. C. D. Higham having been appointed scrutineers, a ballot was taken for the election of President, Vice-Presidents, Council, and Officers for the ensuing year, which resulted in the election of the following gentlemen:—

President.

ROBERT TUCKER.

Vice-Presidents.

ARTHUR H. BAILEY.	ARCHIBALD DAY.
CHARLES JOHN BUNYON, M.A.	HENRY WILLIAM PORTER, B.A.

Council.

MARCUS N. ADLER, M.A.	STEWART HELDER.
ANDREW BADEN.	AUGUSTUS HENDRIKS.
ARTHUR H. BAILEY.	WILLIAM BARWICK HODGE.
GEORGE WILLIAM BERRIDGE.	CHARLES JELLICOE.
SAMUEL BROWN.	JAMES MEIKLE.
CHARLES JOHN BUNYON, M.A.	EDWARD A. NEWTON, M.A.
EDWARD CUTBUSH.	WILLIAM P. PATTISON.
GEORGE CUTCLIFFE.	*ARTHUR PEARSON.
HENRY DEVEREUX DAVENPORT.	HENRY WILLIAM PORTER, B.A.
ARCHIBALD DAY.	HENRY AMBROSE SMITH.
DAVID DEUCHAR.	THOMAS BOND SPRAGUE, M.A.
JOSEPH JOHN DYMOND.	*WILLIAM SMITH.
*ALEXANDER PEARSON FLETCHER.	*JAMES TERRY.
MAJOR-GEN. J. C. HANNYNGTON.	ROBERT TUCKER.
RALPH PRICE HARDY.	JOHN HILL WILLIAMS.

Treasurer.

GEORGE CUTCLIFFE.

Honorary Secretaries.

RALPH PRICE HARDY.	EDWARD A. NEWTON, M.A.

On the motion of Mr. H. AMBROSE SMITH, a cordial vote of thanks was given to the scrutineers.

Mr. ADLARD proposed the election of the following gentlemen, each being an associate of the Institute, as Auditors for the ensuing year:—Mr. Thomas John Searle, Mr. Thomas Emley Young, B.A., and Mr. Charles Daniel Higham.

Mr. R. C. TUCKER seconded the motion, which was at once agreed to.

Mr. GRIMES said he had much pleasure in proposing "That the best thanks of the meeting be given to the retiring President, Vice-Presidents, Council, and Officers, and also to the examiners, for their valuable services during the past year." (Hear, hear.)

Mr. WASBROUGH seconded the resolution, which was cordially and unanimously passed.

The PRESIDENT—"Gentlemen, I can only say for myself that I thank you for the compliment you have paid me. As regards the other members included in this resolution, I will ask Mr. Bailey if he will be good enough to return thanks for himself and them."

Mr. A. H. BAILEY—"It will be wholly unnecessary, Sir, for me to do that; but I may say one word on behalf of the examiners, whose labours are somewhat severe, and not quite so much appreciated as they ought to be." (Hear, hear.)

* New Members.

The PRESIDENT—"That, gentlemen, I think, concludes the business."

Mr. BAILEY—"It seems, Sir, there has been a little omission. In the circular convening the meeting, notice has been given of a proposed alteration in the constitution and laws of the Institute. I think, Sir, the reasons for that alteration are sufficiently obvious. When this Institute—of which you have given so interesting an account to-day—was first founded, I suspect it never occurred to the gentlemen to whom we are indebted for its promotion that it would ever obtain so large a reputation as it has acquired on the Continent and in the United States. But so it is, and many gentlemen in France, Germany, and the United States, as well as in many of the British colonies, have expressed a desire to be admitted to the Institute as ordinary members, and not as mere corresponding members. Two or three of them have been so admitted; but on looking carefully into our constitution and laws there is just a doubt whether they can be duly and properly admitted. With the view of removing that doubt, it was suggested that the laws should be looked through attentively, and two or three verbal alterations made, which will have the effect of making gentlemen not necessarily connected with insurance offices on the Continent and the United States, eligible to become members of the Institute. (Hear, hear.) We believe the alterations proposed will have that effect, and, with your permission, I will move 'That articles 2, 11, and 34, be altered in the manner set out in the printed circular convening the meeting,' namely—

Article 2 of the constitution and laws, line 3, omit the word 'foreign.' Article 11, line 2, omit the words 'and foreign.' Article 11, line 4, omit the words 'each of these classes,' and in lieu thereof substitute 'this class.' Article 34, line 4, omit the words, 'Foreign and.'"

Mr. G. CUTCLIFFE seconded the the motion, which was unanimously adopted.

Mr. SAMUEL BROWN—"Before you finally conclude the business of the day, Sir, I rise with great pleasure to propose a vote of thanks, which I am sure will be cordially supported by every member of the Institute, present as well as absent, I ask you to give a vote of thanks to our esteemed President, not only for the courteous and business-like manner in which he has conducted the proceedings of to-day, but still more for the great services he has rendered to the Institute from the period of its establishment to the present time. (Cheers.) In the very interesting record he has given to-day of the origin of this Institute, he has shown us at what an early period he commenced to forward its interests; and it will be in the recollection of almost every gentleman present that on numerous occasions, owing to his long experience, his high talents, and his devotion to the interests of the Institute, we have had ample reason to be grateful to him for the services he has so freely rendered to us. (Hear, hear.) I therefore propose, with great pleasure, that the thanks of the meeting be presented to the President for his efforts to promote the success and efficiency of the Institute." (Cheers.)

Mr. H. D. DAVENPORT having seconded the motion, it was put to the meeting and carried amid applause.

The PRESIDENT—"Gentlemen, I am very much obliged to you for this unexpected compliment. The flattering manner in which Mr. Brown has spoken of me is, as we all know, due to the kindness of his disposition. As it is expected on these occasions that the President should say something, I thought it might be interesting to you to refer for a few moments to the origin of the Institute which has become so successful. I regret that there are so very few of the original members left, for I am sure they would be as gratified as I am myself at the high position that the Institute has now attained. I thank you very sincerely. (Cheers.) I have to remind the members that the library will be closed as usual during the month of September. That, gentlemen, is the whole of the business.*"

* The above report of the proceedings at the Annual General Meeting is extracted from the *Insurance Record.*

INSTITUTE OF ACTUARIES.

Income and Expenditure for the Year ended 31 *March* 1873.

Dr.

1872-3.	£ s. d.	£ s. d.
Amount of Funds on 31 March 1872, viz. :—		
Mortality Experience Fund	329 8 3	
Messenger Legacy ,,	245 5 1	
Hardy Memorial ,,	212 7 11	
Brown Prize ,,	214 14 10	
General ,,	1151 13 1	
		2153 9 2
Annual Subscriptions, viz. :—		
71 Town Fellows	223 13 0	
26 Country ,,	54 12 0	
124 Town Associates	260 8 0	
79 Country ,,	82 19 0	
1 ,, ,, (in advance)	1 1 0	
		622 13 0
Subscription compounded		21 0 0
Mortality Experience and Life Table Sales		77 14 0
Certificate Fee		5 5 0
Dividend on Messenger Legacy Fund (£211. 1s. 10d. Cons.)	6 4 6	
,, Hardy Memorial ,, (£200 Consols)	5 18 0	
,, Brown Prize ,, (£95. 15s. 5d. 10¼ per-cent East India Stock)	9 17 8	
,, £688. 18s. 2d. Consols	20 6 6	
Interest on Deposit Account	13 2 0	
		55 8 8
ale of Office Furniture		2 0 0
		£2937 9 10

Cr.

1872-3.	£ s. d.	£ s. d.
Rent	100 0 0	
Salaries	126 10 0	
Journal	148 18 11	
Stationery and Printing	98 16 7	
Ordinary Meetings	20 1 8	
General Expenses	66 1 4	
		560 8 6
Tutor	31 10 0	
Library (Books)	1 15 0	
Subscriptions unrecovered	2 2 0	
Cost of Arithmometer presented to Mr. Gray	36 0 0	
		71 7 0
Experience Life Expenses		425 13 2
Amount of Funds, 31 March 1873, viz. :—		
Messenger Legacy Fund	251 9 7	
Hardy Memorial ,,	218 5 11	
Brown Prize ,,	224 12 6	
General ,,	1185 13 2	
		1880 1 2
		£2937 9 10

Balance Sheet, 31 *March* 1873.

	£ s. d.	£ s. d.
Messenger Legacy Fund, viz. :—		
£211. 1s. 10d. Consols, cost	203 17 8	
Unappropriated Dividends	47 11 11	
		251 9 7
Hardy Memorial Fund, viz. :—		
£200 Consols, cost	179 14 6	
Unappropriated Dividends	38 11 5	
		218 5 11
Brown Prize Fund, viz. :—		
£95. 15s. 5d., 10¼ per-cent East India Stock, cost	200 0 0	
Unappropriated Dividends	24 12 6	
		224 12 6
General Fund		1185 13 2
		£1880 1 2

	£ s. d.	£ s. d.
Union Bank Deposit Account		400 0 0
Three per-cent Consols (£1100)		1008 19 3
East India 10¼ per-cent Stock (£95. 15s. 5d.)		200 0 0
Cash, viz. :—		
London and Westminster Bank	247 3 7	
Petty Cash	2 18 4	
		250 1 11
Arrears of Subscriptions		21 0 0
		£1880 1 2

Examined and found correct.

(Signed) { T. E. Young, B.A. / Thos. J. Searle. } *Auditors.*

12 *May* 1873.

INSTITUTE OF ACTUARIES.—SECOND YEAR'S EXAMINATION, 1872.

I.

1. Find the value of ϵ, the base of the Napierean system of logarithms, correct to 4 places of decimals.

2. Given a series of logarithms in arithmetical progression, find the relation in which the corresponding natural numbers stand to one another.

3. If 5 coins be distributed in some unknown way amongst 5 urns, what is the probability that any selected urn will contain 3 coins and no more?

4. Give expressions for the probabilities of the various contingencies arising out of the occurrences of three independent events. Call the events A, B and C, and denote their probabilities by a, b and c respectively.

5. Given $\log 2 = \cdot301030$, $\log 3 = \cdot477121$ and $\log 7 = \cdot845098$, find in what times a sum of money will double and treble itself at 5 per-cent compound interest.

6. Find the value of an annuity certain for n years, whose several payments are $1, 2, 3, \ldots n$.

7. Give some account of the difference between the materials used in the construction of Davies's and Morgan's "Equitable" Tables.

8. State briefly the basis upon which the Government Annuity Tables (Finlaison, 1860) are founded.

9. Find an expression for the average age at death of l_x persons.

10. Distinguish between the average after-lifetime and the most probable after-lifetime.

11. Give a practical method of forming a Table of the values of $\log \frac{1}{2}(p_x^{-1}-1)(p_y^{-1}+1)$.

12. Explain the construction of the several columns of a single life Commutation Table.

13. Given the annual premium and the rate of interest, show how to find the value of £1 payable at the end of the year of death.

II.

14. Assuming the values of δ and $\bar{a}_x$ to be known, find the value of $\bar{A}_x$, the present value of a continuous assurance of £1 payable at the instant of death.

15. Find an expression for the value of a life-annuity, when the decrement of life at each age (x) is equal to x.

16. Investigate a formula by which a table of policy-values $({}_nV_x)$ may be formed successively.

17. Show, under what circumstances, the value of a policy at the end of the $(n+1)^{\text{th}}$ year, $({}_{n+1}V_x)$, will be greater than the value at the end of the n^{th} year with the premium just paid.

18. Assuming that the lives enter proportionately to the numbers shown in Table of Mortality, find the annual premium that may be charged uniformly to all entrants from age (x) to age $(x+n)$ inclusive for ordinary whole term assurances of £1.

19. Investigate an expression for the value of A^1_{xy}.

20. Determine $A^{1,3}_{x:yz}$, the present value of £1 payable upon (x) failing either the first or last of three lives (x), (y) and (z).

21. A copyhold estate is held on a single life, subject to a fine for

renewal on the death of each nominee. Find the value of the fines in perpetuity.

22. An assurance office guarantees reversionary bonus additions to the sums assured under its policies, at the rate of 1 per-cent per annum on the sum originally assured for each premium paid: what is the average amount of bonus that lives entering at age (x) will receive?

23. For what amount should a policy, free from future payment of premium, be granted in consideration of the surrender of an existing assurance? Assume all values to be net.

24. Prove the following equation:

$$ {}_nV_x=1-(1-V_x)(1-V_{x+1}) \ldots . (1-V_{x+n-1}). $$

25. A (who is tenant for life in possession of an estate) agrees to join with B (who is tenant for life in remainder, expectant on the death of A) in exercising their joint power of raising money on the fee simple of the estate. The money so raised is to be applied solely for the benefit of B, who agrees to compensate A for the loss of income he will sustain in having to keep down the interest on the sum so raised. How much, out of every £1 so raised, should be reserved for A?

THIRD YEAR'S EXAMINATION, 1872.

I.

1. Explain briefly how the rate of mortality which has been experienced by a Life Assurance Company is to be deduced from the facts contained in its registers.

2. Give some account of the method of graduation adopted in the construction of the Institute of Actuaries' mortality tables.

3. Point out the difference between and compare the merits of the Seventeen Offices' Experience and the Institute of Actuaries' mortality tables.

4. Describe the method of distribution of profits introduced by Mr. Jellicoe.

What is required to make it work equitably?

5. Describe the method of distribution of profits introduced by Mr. Homans, and called by him his "Contribution Method."

6. Does the maxim "*caveat emptor*" apply to the contract of Life Assurance?

7. What is meant by a policy being in the order and disposition of the Bankrupt with the consent of the true owner? And explain whether any alteration in the Law has been proposed or effected within the last 4 years.

8. State the effect of the Life Insurance clauses in the Married Women's Property Act 1870.

9. What is meant by "novation"?

10. The liability of Shareholders in Life Offices is not usually limited in England either by charter of incorporation or registration as limited companies under the Companies Act 1862; explain the manner in which in such unlimited companies a limitation of the Shareholders' liability is sought, and state the limit of the limitation in a Liquidation.

11. What are the principal subjects on which statistics are annually published in the Statistical Abstract for the United Kingdom?

12. Write a valuation report of a Life Assurance Company framed so as to afford in the shortest compass the information required for understanding its real position.

13. What have been the principles adopted in the changes that have been made since 1842 in the customs duties of the United Kingdom?

II.

14. Admitting that it is expedient to obtain part of the revenue of the country by direct taxation, Do you think that the present income tax best answers this purpose? and if not—What alteration would you suggest?

15. Point out the difference in the contract between the Government and the Fundholder: and between a Railway Company and its Debentureholders.

16. Explain the constitution of the Bank of England.

17. State shortly the opinions entertained by Lord Overstone and the late Mr. Tooke in reference to the separation of the Issue and Banking departments of the Bank of England.

18. Name any classes of Investments that you consider to be specially suited for the investments of a Life Assurance Company, and give your reasons.

19. How and why do the fluctuations in the rate of interest on mortgages and similar securities differ from those in the Bank rate of discount?

20. If a Company holds British or Foreign Government Securities as part of its investments, describe the different methods which may be followed in valuing them at a periodical valuation of the Company's Assets and Liabilities. Which do you consider the best method, and why?

21. State the principal differences between a "cash account" and a "revenue account," and point out why the latter is more suitable than the former for the accounts of Life Insurance Companies. Describe the connection between the "revenue account" and the "balance sheet" of the 1st and 2nd schedules of the Life Assurance Companies Act 1870.

The Candidate having completed and handed in his answers to the above questions will be allowed the use of any books he may desire in answering the following questions:—

22. What is the market value of the reversion to £1000 Consols receivable at the death of a lady age 65 if a gentlemen age 30 shall survive her?

23. A, age 35, is entitled to a reversionary life interest in an estate of ample value, provided his uncle age 70 die without leaving male issue—his wife being 65 and in good health. What reversionary annuity should A grant in consideration of a present advance of £1000? How would you provide for the risk of A proceeding to a tropical climate?

24. What, in your opinion, is the value of the next presentation to a living producing £600 a year net, with a good house, the age of the present incumbent being 72?

25. A loan is raised by the issue of £5,000,000 5 per cent stock at the price of 85; and the stock is to be paid off at par by annual drawings by means of an accumulative sinking fund of £200,000 per annum. In how many years will the loan be paid off? And what rate of interest is virtually paid by the borrower?

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C. & E. Layton, Printers, 150, Fleet Street; and 12, Gough Square.

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